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ABSOLUTE MEASUREMENT OF CAPACITANCE BY MAXWELL'S METHOD

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ABSTRACT

The absolute measurement of capacitance by Maxwell's method is based on the assumption that certain conditions are fulfilled by the experimental apparatus. While there are at least seven of these conditions, the most important is that the galvanometer correctly integrates the current. When large capacitances are measured, a galvanometer is required in which the current through the coil does not affect the magnetism of the permanent magnet. By placing the coil in a symmetrical position with respect to the magnet this condition can be fulfilled in most galvanometers. However, this adjustment is greatly simplified by a proper design of the galvanometer.

The method requires that a condenser shall be charged and discharged at a known rate. To accomplish this, contacts of platinum dipping in mercury cups have been placed on the prongs of a tuning fork driven by an electron tube. This permits the Maxwell bridge to be balanced with the same ease and accuracy as a Wheatstone bridge. However, it is necessary to determine the frequency of the fork at the time of measurement. For this purpose the fork is compared with a freely vibrating pendulum by a method which gives an accuracy of a part in a million with observations which extend over a few minutes' time.

The possibility of introducing systematic errors has been guarded against by employing two independent bridges in which the factors have been varied as much as possible. Results are given showing the accuracy that can be attained.

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I. INTRODUCTION

The absolute measurement of capacitance has been carried out more often by Maxwell's method than by any other. More than 20 years ago there was published [1] a description of the method as then used in the inductance and capacitance laboratory of this bureau. Since then a number of modifications have been made and the theory of the method considerably extended. The present paper describes the apparatus as now used, discusses the assumptions on which the theory is based, and gives experimental data to show the accuracy which can be attained.

The method was first described by Maxwell,[2] but there is no record that it was ever used by him. It was first used by J. J. Thomson,[3] who derived an accurate formula for computing the capacitance. Some important theoretical and experimental facts have been recorded by those who have used the method. Diesselhorst [4] showed the importance of preventing oscillatory discharge. Glazebrook [5] found that the method was not suitable for measuring any condenser which had absorption. Dittenberger and Grüneisen [6] were unable to detect any change in capacitance of an air condenser when the number of charges and discharges per second was varied from 18 to 94. Giebe [7] derived a formula giving the correction for imperfect insulation of the charge and discharge apparatus. Orlich [8] indicated the importance of grounding the bridge.

This paper has been prepared not only to describe the method as used but also to point out the difficulties which may arise with apparatus which is not well adapted for this experiment. This purpose can best be accomplished by a careful theoretical consideration of the underlying physical principles. Hence the method is fully described, the formula for computing the capacitance derived, and the sources of error discussed.

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[1] Rosa and Grover, The Absolute Measurement of Capacity. Bull. of Bureau of Standards, 1, p. 154; 1905.

[2] Maxwell, Electricity and Magnetism (1st ed.), 2, art. 771; 1873.

[3] Thomson, J. J., "On the determination of the number of electrostatic units in the electromagnetic units of electricity." Phil. Mag., 174, p. 707; 1883.

[4] Diesselhorst, H., Zu Maxwell's Methode der Absoluten Messung von Kapazitaten. Ann. d. Physik No. 324; p. 382; 1906.

[5] Glazebrook, R. T., "On the air condensers of the British association." Electrician, 25, p. 616; 1890.

[6] Dittenberger and Grüneisen, Thatigkeitbericht der Physikalisch Technischen Reichsanstalt. Zs. für Instrk., 21, p. 111; 1901.

[7] Giebe, E., Normal-Luftkondensatoren und ihre absolute Messung. Zs. für Instrk., 29, p. 269; 1909.

[8] Orlich, E., Kapazität und Induktivität, p. 212; 1909.

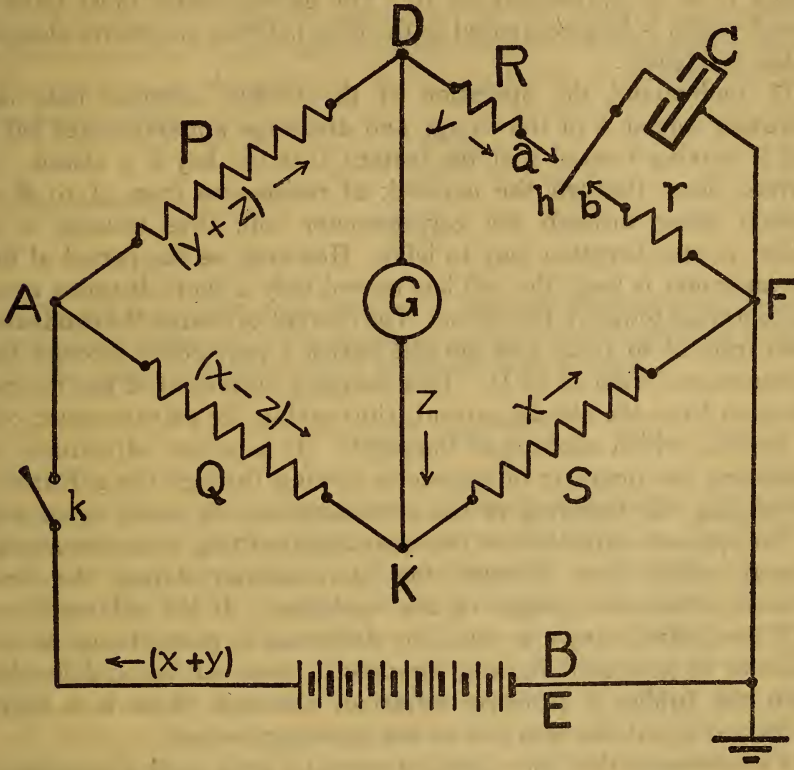

FIG. 1.—*Diagram of connections of a Maxwell bridge for the absolute measurement of capacitance*

The capacitance to be measured is represented by C.

The resistances are represented by P, Q, R, S, B, G, and r.

The instantaneous values of the currents are represented by x, y, z, and combinations of these letters. The arrows indicate the directions of the currents at the instant considered in writing the equations.

The charge and discharge apparatus is indicated by a, h, b.

II. DESCRIPTION OF MAXWELL'S BRIDGE FOR THE ABSOLUTE MEASUREMENT OF CAPACITANCE

Maxwell's capacitance bridge is a Wheatstone network in which one resistance arm is replaced by an apparatus for charging and discharging a condenser at a known rate. The electrical connections are shown diagrammatically in Figure 1.

The three resistance arms are indicated by P, Q, S. In the fourth arm is the condenser C, together with the charge and discharge apparatus a, h, b. For generality there is indicated a resistance R in the fourth arm. This ordinarily is the resistance of the condenser leads only.

Between the points A F there is a battery of electromotive force E. The total resistance of the battery and leads to the points A and F is represented by B.

The resistance of the galvanometer and of the leads between the points D K is represented by G. The galvanometer must have a period which is long compared to the time between successive charges of the condenser.

To understand the operation of the bridge, consider that the vibrating tongue h of the charge and discharge apparatus has left b and is moving toward a at the instant that the key k is closed. A current flows through the network of resistances from A to F, a portion going through the galvanometer coil, thus causing it to deflect in one direction (say to left). However, as the period of the galvanometer is long, the coil has moved only a short distance when the vibrating tongue h touches a. The current to charge the condenser flows from A to D by two parallel paths, a part going through the galvanometer from K to D. This charging current is in the reverse direction from the steady current, thus giving the galvanometer coil an impulse which sends it to the right. By a proper adjustment of resistance the quantity of electricity flowing through the galvanometer during the charging of the condenser can be made equal and in the opposite direction to the quantity resulting from the steady current which flows through the galvanometer during the time between successive charges of the condenser. If the galvanometer is of the ballistic type, so that the deflection is proportional to the quantity of electricity passing through it, there will be no deflection when the bridge is properly adjusted, although there is a slight oscillation about the zero due to the pulsating current.

It is necessary that the vibrating tongue h shall touch b in order to discharge the condenser. The energy of this discharge must be dissipated in the resistance r. However, the discharge current has no influence on the galvanometer deflection.

It is customary to ground the bridge at F (as shown in fig. 1). This insures that all capacitances to earth shall be definite. However, experiments in this laboratory with large condensers show that, if all parts of the bridge are well insulated, the measured value of the capacitance is not affected by the grounding of the bridge.

III. DERIVATION OF THE FORMULA FOR THE CAPACITANCE

The formula for the capacitance as given by Maxwell[9] was only an approximation. A more complete formula was derived by J. J. Thomson.[10] He assumed that the galvanometer integrates the current correctly and that, when a balance is obtained, the galvanometer coil is stationary. The following derivation uses the first of these assumptions but avoids the second.

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[9] See footnote 2, p. 488.

[10] See footnote 3, p. 488.

In Figure 1 let the values of the currents in the arms $K F$, $D F$, and $D K$ at the time t be represented by x, y, and z, respectively, the arrows indicating the positive direction of the current. The instant that h touches a is taken as zero time. As the phenomena is repeated after an interval T, the value of t will always lie between zero and T. The quantity of electricity q on the condenser C at time t is given by the equation

$$q=\int_0^t ydt$$

Differentiating

$$\frac{dq}{dt}=y \tag{1}$$

In the galvanometer arm there is an induced electromotive force, e, due to two causes, (1) the changing current in an inductive coil and (2) the motion of the coil in a magnetic field. Then

$$e=L\frac{dz}{dt}+gH\frac{d\theta}{dt} \tag{2}$$

where L is the self-inductance of the galvanometer, g the area of galvanometer coil multiplied by the number of turns, and H the value of the magnetic field at the coil when it is in position θ.

When the key k is closed there is usually a considerable initial deflection of the galvanometer. If the bridge balance is correct, this disappears in about a half period of the galvanometer. From then on a steady state exists. Hence, it is only necessary to consider one cycle of the charge and discharge apparatus. This cycle will start at the time $t=0$ when h touches a. At $t=t_1$, h leaves a, thus disconnecting C from the bridge. At $t=T$ the cycle is completed and h again touches a.

Applying Kirchoff's laws to the three circuits $D F K$, $D K A$, and $A K F$, there result the three following equations which hold during the interval from $t=0$ to $t=t_1$.

$$Ry+\frac{q}{C}-Sx-Gz=L\frac{dz}{dt}+gH\frac{d\theta}{dt} \tag{3}$$

$$Gz-Q(x-z)+P(y+z)=-L\frac{dz}{dt}-gH\frac{d\theta}{dt} \tag{4}$$

$$Q(x-z)+Sx+B(x+y)=E \tag{5}$$

Rearranging these equations with respect to the variables x, y, z, and θ, letting

$$S+Q+B=S'$$

$$P+Q+G=P'$$

and representing derivatives with respect to time by a dot placed over the variables, these equations become

$$Sx-Ry+Gz+L\dot{z}=\frac{q}{C}-gH\dot{\theta} \tag{6}$$

$$-Qx+Py+P'z+L\dot{z}=-gH\dot{\theta} \tag{7}$$

$$S'x+By-Qz=E \tag{8}$$

In the interval between $t=t_1$ and $t=T$, R is infinite and y is zero. Hence in this interval equation (6) is nonexistent and (7) and (8) are modified in that the terms in y disappear.

Eliminating x between equations (7) and (8) gives

$$(PS'+BQ)y+(P'S'-Q^2)z+LS'\dot{z}=EQ-gHS'\dot{\theta} \tag{9}$$

for the first part of the cycle between $t=0$ and $t=t_1$. The equivalent equation for the second part of the cycle between $t=t_1$ and $t=T$ differs only in that the term in y disappears. The condition under which a balance of the bridge is obtained is that $\int_0^T zdt=0$. This condition can be applied to equation (9) and its equivalent for the second part of the cycle by multiplying each by dt and integrating the first from 0 to t_1 and the second from t_1 to T. The following equations result:

$$(PS'+BQ)\int_0^{t_1} ydt+(P'S'-Q^2)\int_0^{t_1} zdt+LS'(z_1-z_0) \\ =EQt_1-gHS'(\theta_1-\theta_0) \tag{10}$$

$$(P'S'-Q^2)\int_{t_1}^{T} zdt+LS'(z_T-z_1)=EQ(T-t_1)-gHS'(\theta_T-\theta_1) \tag{11}$$

In these equations z_0, z_1, and z_T represent the values of z at times $t=0, t=t_1$, and $t=T$; while θ_0, θ_1, θ_T represent the values of θ at these times. Add equations (10) and (11)

$$(PS'+BQ\int_0^{t} ydt+(P'S'-Q^2)\int_0^{T} zdt+LS'(z_T-z_0) \\ =EQT-gHS'(\theta_T-\theta_0) \tag{12}$$

But when the bridge is balanced so that $\int_0^T zdt=0$, a cyclic condition is set up in which both the current through the coil and its displacement have the same values at the beginning and end of the cycle; that is, $z_0=z_T$ and $\theta_0=\theta_T$. Also, if the condenser is completely discharged at the beginning of the cycle, the quantity of electricity, q_1, on the condenser at the instant t_1 is $\int_0^t y dt$. Hence, equation (12) becomes

$$(PS'+BQ)q_1=EQT \tag{13}$$

This value of q_1 has been obtained by the use of integrals obtained from equations (7) and (8) only. Another independent value can be obtained from equation (6) simply by substituting the values of the variables at time t_1. However, this gives the value of q_1 in terms of four other variables. An equivalent but more satisfactory method of obtaining q_1 is to treat equations (6), (7), and (8) as simultaneous differential equations in q, x, z, and θ from which an equation involving only two variables can be derived. Then the values of these two variables and their derivatives at time t_1 gives immediately the desired result.

Using the symbolic form of notation and observing that $Dq=y$, equations (6), (7), and (8) become

$$-(RD+1/C)q+Sx+(LD+G)z=-gHD\theta \tag{14}$$

$$PDq-Qx+(LD+P')z=-gHD\theta \tag{15}$$

$$BDq+S'x-Qz=E \tag{16}$$

These equations can be solved by determinants [11] for q.

$$\begin{vmatrix} -\left(RD+\frac{1}{C}\right) & S & LD+G \\ PD & -Q & LD+P' \\ BD & S' & -Q \end{vmatrix} q = \begin{vmatrix} -gHD\theta & S & LD+G \\ -gHD\theta & -Q & LD+P' \\ E & S' & -Q \end{vmatrix} \tag{17}$$

Expanding and collecting terms

$$\Big[L\Big\{S'(P+R)+B(S+Q)\Big\}D^2+\Big\{P(S'G+SQ)+B(P'S+GQ)+ R(P'S'-Q^2)\Big\}D+\frac{1}{C}(P'S'-Q^2)\Big]q=E(P'S+GQ)+ gH(P'S'-Q^2-SQ-S'G)D\theta \tag{18}$$

Instead of attempting to solve this equation, the assumption will be made that the experimental apparatus is adjusted until all three of the derivatives are zero at time t_1 when the vibrator breaks contact. Then at this instant the terms involving the velocity of the galvanometer coil ($D\theta$), the current into the condenser (Dq), and the rate of change of the current (D^2q) shall all be negligibly small. Under these conditions

$$q_1=\frac{EC(P'S+GQ)}{P'S'-Q^2} \tag{19}$$

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[11] Johnson, Differential Equations, p. 262.

Equating this to the value of q_1 given by equation (13) and solving for C

$$C=\frac{QT(P'S'-Q^2)}{(PS'+BQ)(P'S+GQ)}=\frac{QT}{PS}\frac{1-\frac{Q^2}{P'S'}}{\left(1+\frac{BQ}{PS'}\right)\left(1+\frac{GQ}{P'S}\right)}=\frac{QTF}{PS} \qquad (20)$$

This result is identical with the one obtained by J. J. Thomson.

The derivation of equation (20) is based on the assumption that the following conditions have been fulfilled in the experimental method:

1. The battery has been connected to the bridge for a number of cycles until a steady state has been reached. Under this condition the current in, and motion of, the galvanometer coil is the same in the cycle under observation as in the preceding and following cycles.
2. The condenser is completely discharged at the beginning of the cycle.
3. During each cycle the time that the vibrator *h* is in contact with the point *a* is sufficient to allow the condenser to become completely charged so that both the first and second time derivatives of the charge are negligibly small. Hence, in time t_1 the charge on the condenser must reach a steady value which is very near the value it would reach in infinite time.
4. The vibrator *h* breaks contact with the point *a* at the instant that the velocity of the galvanometer coil is zero. This requires that the vibrator break shall be correctly timed.
5. The resistance between the contacts *a* and *h* of the vibrator becomes infinite as soon as *h* leaves *a*. This requires that the insulation of the vibrator shall be perfect.
6. The inductance of the galvanometer coil is independent of the current through the coil and of the position of the coil.
7. The time integral of the current through the galvanometer is zero for the cycle.

IV. DISCUSSION OF THE ERRORS INTRODUCED BY FAILURE TO MEET THE ASSUMPTIONS

The conditions which have just been enumerated must be so well met by the experimental apparatus that in no case will a significant error be introduced. In order to design suitable apparatus, it is necessary to study the fundamental relationships between the apparatus which can be employed and the conditions which have been established. Also there should be a theoretical and experimental study of the error which will be introduced by a failure to completely realize each one of the conditions. The following discussion considers the errors which may be introduced.

1. THE STEADY STATE

The position of the vibrator *h* at the instant the battery key is closed is a matter of chance. Should the key be closed at the instant that *h* touches *a*, then the current would have nearly the same value in the first cycle as in those following. However, the galvanometer motion would be quite different from that of the following cycles, since at the beginning of the first cycle the velocity of the galvanometer coil is zero, which is not the case in later cycles. The result is a deflection in the direction which would be produced by the charging current of the condenser. If the key is closed a little later in the cycle, the deflection will be even larger. The maximum deflection in this direction will occur if the key is closed just before *h* leaves *a*.

If the key is closed just after *h* leaves *a*, then the coil will be given, by the steady current, a velocity which at the beginning of the next cycle will be greater than normal. Hence, the initial deflection will be in the opposite direction from that produced by charging the condenser. However, this will be a relatively small deflection. As the closing is made still later in the cycle, the initial deflection decreases. When the integral of the steady current for the remaining part of the cycle is not sufficient to give the galvanometer coil a velocity equal to that which it normally has at the beginning of a cycle, then the initial deflection is in the direction produced by charging the condenser.

From the above discussion it follows that the deflection is more likely to occur in the direction produced by the current charging the condenser. The relative number in the two directions will depend in part on the portion of a cycle that *h* is in contact with *a*. Some tests with apparatus used in this laboratory gave about 80 per cent of the initial deflections in the direction produced by the current charging the condenser.

In whatever part of the cycle the key is closed it is merely necessary to delay the taking of a reading until the current through the galvanometer and the resulting motion of the galvanometer coil have reached a steady state. The time required for the motion of the galvanometer coil to become steady will largely depend on the constants of the galvanometer. To make this time small, the period should be short and the damping critical. However, as some of the other conditions demand a long period, it is not generally feasible to make this time less than 30 seconds. In this time, even if the condenser is charged and discharged only 100 times per second, there will be 3,000 cycles of the current, which is much more than enough for the current through the galvanometer to reach a steady state. Hence in starting an observation it is only necessary to wait until the galvanometer deflection becomes steady to be assured that a steady state has been reached.

2. COMPLETENESS OF DISCHARGE

The vibrator *h* must be in contact with the point *b* for a time sufficient to insure that the charge left on the condenser is negligibly small. If the inductance and resistance of the discharge circuit are known, then the time for the residual charge to become less than one-millionth of the original charge can be computed. This time is shortest when the resistance is just sufficient to prevent an oscillatory discharge. Hence, it is well to insert in the discharge circuit between *b* and *F* a resistance sufficient to insure that the discharge is aperiodic but which is not so large as to prevent a complete discharge of the condenser.

To test for completeness of discharge the resistance in the discharge circuit is suddenly increased to several hundred ohms. If there is no appreciable change of the galvanometer deflection the discharge is complete.

3. COMPLETENESS OF CHARGE

In charging a condenser through a resistance theory shows that an infinite time is required for the charge to become complete. However, in a very short interval of time (often only a few ten-thousandths of a second), the charge will become so nearly the value it would have in infinite time that experimentally the one charge can not be distinguished from the other. In using a Maxwell bridge it is important that the contact shall be made for a sufficient time to allow the charge to become experimentally equivalent to the charge at infinite time.

The condenser is said to be completely charged at time *t* when the difference between the charge q_t at time *t* and the charge q_∞ at infinite time is relatively less than the allowable experimental error. Stated as an inequality, the condenser is completely charged when

$$\frac{q_\infty - q_t}{q_\infty} < \tau \tag{21}$$

where τ is the allowable experimental error. The left-hand member of the above inequality will be called the incompleteness of charge.

In order to determine a value for the incompleteness of charge, equation (18) can be solved under the condition that $D\theta$ is zero. To simplify the algebraic work, the following substitution will be made:

$$\begin{aligned}
a &= S'(P+R) + B\ (S+Q)\\
\beta &= P(S'G+SQ) + B\ (P'S+GQ) + R(P'S'-Q^2)\\
k &= P'S'-Q^2 = P'(S+B) + Q(P+G)\\
b &= P'S+GQ
\end{aligned}$$

It should be noted that all of the above quantities are positive functions of the resistances. Hence, equation (18) becomes

$$\left[La D^2+\beta D+\frac{k}{C}\right] q=Eb \tag{22}$$

Solving this differential equation

$$q=A_1 e^{\lambda_1 t}+A_2 e^{\lambda_2 t}+\frac{ECb}{k} \tag{23}$$

where A_1 and A_2 are constants to be determined from the initial conditions and λ_1 and λ_2 are found by equating to zero the coefficient of q in (22) and solving for D as though it were an algebraic variable. Then

$$\lambda_1=-\frac{\beta}{2La}+\sqrt{\frac{\beta^2}{4L^2a^2}-\frac{k}{LCa}} \tag{24}$$

$$\lambda_2=-\frac{\beta}{2La}-\sqrt{\frac{\beta^2}{4L^2a^2}-\frac{k}{LCa}} \tag{25}$$

The values of A_1 and A_2 can be determined from the conditions that, when $t=0$, $q=0$ and $y=\frac{dq}{dt}=\frac{E}{P+R+B}$.

Hence

$$A_1+A_2=-\frac{ECb}{k} \tag{26}$$

and

$$A_1\lambda_1+A_2\lambda_2=\frac{E}{P+R+B} \tag{27}$$

Solving

$$A_1=\frac{E}{\lambda_1-\lambda_2}\left[\frac{Cb\lambda_2}{k}+\frac{1}{P+R+B}\right] \tag{28}$$

$$A_2=-\frac{E}{\lambda_1-\lambda_2}\left[\frac{Cb\lambda_1}{k}+\frac{1}{P+R+B}\right] \tag{29}$$

The values of λ_1 and λ_2 will be complex numbers if $\frac{\beta^2}{4L^2a^2}<\frac{k}{LCa}$, otherwise they will be real. In case they are complex, the real part will be negative; in case they are real, the values of both are negative. Hence, if $t=\infty$, both the exponentials in (23) vanish under all conditions. Therefore

$$q_\infty=\frac{ECb}{k} \tag{30}$$

while at time $t=t_1$

$$q_t=\frac{ECb}{k}+A_1 e^{\lambda_1 t_1}+A_2 e^{\lambda_2 t_1} \tag{31}$$

Substituting in (21), the incompleteness of charge is

$$\frac{q_\infty - q_t}{q_\infty} = -\frac{k}{ECb}\left(A_1 e^{\lambda_1 t_1} + A_2 e^{\lambda_2 t_1}\right) \tag{32}$$

The values of the A's and λ's are given in terms of the bridge constants in equations (24), (25), (28), and (29). The substitution in any particular case is perfectly definite but somewhat laborious.

For most work it suffices to assume that L is zero. While the value of the incompleteness of charge can be obtained by substituting $L=0$ in (32), it is simpler to solve equation (22) as a first-order differential equation. Then

$$q = \frac{ECb}{k} + A_1 e^{-\frac{kt_1}{C\beta}} \tag{33}$$

From the condition that when $t=0$, $q=0$, it follows that

$$\frac{q_\infty - q_t}{q_\infty} = e^{-\frac{kt_1}{C\beta}} \tag{34}$$

By consulting tables of exponentials or by taking the logarithm of (34) it can be shown that in order for the incompleteness of charge to be less than a part in a million

$$\frac{kt_1}{C\beta} \geqq 14$$

Since a knowledge of this is required only to insure that it is sufficiently large, an approximate value is entirely satisfactory. Hence the following approximations can be made:

$$t_1 = \frac{T}{2} \qquad C = \frac{QT}{PS}$$

$$k = (P+Q+G)S$$

$$\beta = PS(Q+G)$$

Hence

$$\frac{kt_1}{C\beta} = \frac{S(P+Q+G)}{2Q(Q+G)} \tag{35}$$

With any given set-up one can be assured that the incompleteness of charge is not more than a part in a million if, on substituting the resistances in (35), the value of the fraction is greater than 14.

While the above computation is useful in many cases, a more satisfactory method of knowing when the charge is complete is to make an experimental test. If, when the bridge is balanced, a sudden increase of R from a value near zero to a value approximately as large as Q produces only a negligible effect on the bridge balance, then with R reduced to zero the incompleteness of charge must be negligible.

4. EFFECT OF INCORRECT TIMING OF THE VIBRATOR BREAK

The vibrator h should leave the contact a at the instant, t'_0, that the velocity of the galvanometer coil is zero. When this condition is realized, the vibrator break is said to be correctly timed. If the break occurs at a time t', either slightly before or after t'_0, then the incorrectness in timing is $t'-t'_0$.

In order to determine the effect on the computed capacitance caused by incorrect timing, it is necessary to determine an expression for the velocity of the coil which can be inserted in equation (18). This expression can be obtained by solving the galvanometer equation, namely,

$$K\frac{d^2\theta}{dt^2}+M\frac{d\theta}{dt}+U\theta=gHz \qquad (36)$$

where

K is the moment of inertia of the galvanometer coil,
M is the damping,
U is the restoring torque of the suspension,
H is the magnetic field in which the coil swings,
g is the product of the area of the coil and the number of turns,
θ is the angular deflection of the coil at the time t,
z is the current through the coil at the time t.

In the portion of the cycle under consideration, θ is negligibly small while z is practically constant and equal to $\frac{EQ}{P'S'-Q^2}$. Making these substitutions and using the symbolic form for the derivative.

$$\left(D+\frac{M}{K}\right)\dot{\theta}=\frac{gHEQ}{K(P'S'-Q^2)} \qquad (37)$$

Integrating this equation and determining the constant of integration from the condition that $\dot{\theta}=0$ when $t=t'_0$,

$$\dot{\theta}=\frac{gHEQ}{M(P'S'-Q^2)}\left[1-e^{-\frac{M}{K}(t-t'_0)}\right] \qquad (38)$$

As the damping is seldom more than critical, an idea of the maximum value of the exponential can be obtained by substituting in (38) the condition for critical damping; that is,

$$M^2=4KU \qquad (39)$$

Substituting this value of M and observing that the free period of the galvanometer, T_g, is $2\pi\sqrt{\frac{K}{U}}$, the exponent in (38) becomes

$$-\frac{4\pi\,(t-t'_0)}{T_g} \qquad (40)$$

Since the period of the galvanometer is always long relative to the largest possible value of $(t-t'_0)$, the value of the exponent is always very small. Hence, the exponential in (38) can be expanded in a series. Making this expansion, substituting the values of M on the assumption of critical damping, and retaining only the first power of $\frac{t-t'_0}{T_g}$

$$\dot{\theta}=\frac{gHEQ\,(t-t'_0)}{K\,(P'S'-Q^2)} \tag{41}$$

Substituting the value of $\dot{\theta}$ from (41) in (18), assuming that the term containing L can be neglected and using the equivalents given in the preceding section, and the additional one that

$$h=P'S'-Q^2-SQ-S'G=PS'+QB$$

there results the differential equation

$$\left[\beta D+\frac{k}{C}\right]q=Eb+\frac{Eg^2H^2Qh\,(t-t'_0)}{Kk} \tag{42}$$

Solving this equation

$$q=\frac{EbC}{k}+\frac{Eg^2H^2QhC}{K\,k^2}\left(t-t'_0-\frac{C\beta}{k}\right)+Ae^{-\frac{kt}{C\beta}} \tag{43}$$

where A is the constant of integration. The value of A can be determined from the condition that $q=0$ when $t=0$. However, as A is finite and the exponential infinitesimally small throughout the time interval under consideration, the entire term can be neglected. Also as a first approximation

$$\frac{C\beta}{k}=\frac{Q\,(Q+G)\,T}{S\,P'} \tag{44}$$

Making these substitutions

$$q=\frac{EbC}{k}+\frac{Eg^2H^2QhC}{Kk^2}\left(t-t'_0-\frac{Q\,(Q+G)\,T}{SP'}\right) \tag{45}$$

The last term in the parentheses is always a small fraction of T, and hence can be neglected, since $t-t'_0$ may be a large fraction of T.

Equation (45) holds throughout the region during which the vibrator might break contact. If this break occurs at time t', when $q=q_1$, then the value of q as obtained from (45) can be equated against the value from (13). It follows that

$$\frac{EQT}{h}=\frac{EbC}{k}+\frac{Eg^2H^2QhC(t-t'_0)}{Kk^2} \tag{46}$$

Solving for C

$$C=\frac{kQT}{bh\left[1+\frac{Qg^2H^2h(t'-t'_0)}{Kbk}\right]} \tag{47}$$

In this equation both the electrical, magnetic, and mechanical quantities must be measured in units of the same system, preferably the cgs.

The galvanometer constants, g, H, and K, are not readily measured. However, in the case of galvanometers which are not wound on a metal frame, these constants can be replaced by others which are easily measured.[12] In such galvanometers the value of M, which includes not only mechanical damping but also that caused by eddy currents, is sufficiently small so that it may be neglected. Then on open circuit

$$K\frac{d^2\theta}{dt^2}+U\theta=0 \tag{48}$$

Solving by assuming a simple harmonic vibration

$$\theta=A\sin\omega t$$

and substituting in (48) it follows that

$$\omega^2K=U \tag{49}$$

where $\omega=2\pi/T_g$; T_g being the complete period of the galvanometer.

If then the external critical damping resistance R_c is determined, the equation

$$K\frac{d^2\theta}{dt^2}+\frac{g^2H^2}{R_c+G}\frac{d\theta}{dt}+U\theta=0 \tag{50}$$

will have equal roots (the condition for critical damping) if

$$\frac{g^4H^4}{(R_c+G)^2}=4KU \tag{51}$$

Substituting the value of U from (49) in (51), extracting the square root and rearranging

$$\frac{g^2H^2}{K}=2\omega(R_c+G)=\frac{4\pi(R_c+G)}{T_g} \tag{52}$$

If this value and the values of b, h, j, and k are substituted in (47) then since the quantity in the denominator in brackets is only slightly different from unity

$$C=\frac{kQT}{bh}\left[1-\frac{4\pi Qh(R_c+G)(t'-t'_0)}{bkT_g}\right] \tag{53}$$

Had no account been taken of the incorrectness of timing, the computed capacitance $C\alpha$ would have been

$$C\alpha=\frac{QTF}{PS}=\frac{kQT}{bh}$$

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[12] For a complete discussion of the galvanometer equations see Wenner, Bull. Bureau of Standards, **13**, p. 211.

Hence the error which incorrect timing may introduce is

$$\frac{C\alpha - C}{C\alpha} = \frac{4\pi Qh(R_c + G)(t' - t'_0)}{bkT_g} \tag{54}$$

If $P = Q$ and G is small enough to be neglected, k differs but little from $2h$, so that approximately $4Qh/bk = C/T$

$$\frac{C\alpha - C}{C} = \frac{\pi CR_c(t' - t'_0)}{T_g T} \tag{55}$$

The error due to a lack of correct timing is zero if $t' = t'_0$. This condition can not always be met in practice. However, by choosing a galvanometer in which R_c is small and T_g is large, no measurable error from this cause will be introduced even if t' differs appreciably from t'_0.

5. IMPERFECT INSULATION OF THE VIBRATOR

The formula for the capacitance was derived on the assumption that, when the vibrator breaks contact, the resistance becomes infinite. Such a condition can never be fully realized in practice. Hence, it is important either to outline a method which will show when the insulation is sufficiently high or to derive a formula which will hold for imperfect insulation.

It is not difficult to modify the solution already given to include a leakage either across the vibrator, or across the condenser, or both. However, such a solution has little practical value. An insulation resistance varies greatly with changes either in humidity or in temperature. In an ordinary laboratory it is not unusual to have the insulation resistance of an instrument change by a factor of 10 in a few hours. Hence, it is not desirable to use apparatus with so poor insulation that a correction on its account must be applied.

It is important to know that the insulation resistance of all parts of the apparatus is so high that the leakage currents do not affect the measured capacitance. The two places where leakage is most likely to occur are across the vibrator and across the condenser. To test whether the leakage across the vibrator is important, the vibrator should be stopped with the tongue h between the two contacts. Then with S infinite, P large, and Q small, the only current which flows through the galvanometer is the leakage current across the commutator. If no deflection is observable, then the leakage current across the vibrator produces no effect on the measured capacitance. Moreover, when the bridge is used with equal arms so that $P = Q$, one is assured that the insulation resistance must decrease by at least a factor of 2 from what it was during the test to produce an appreciable error.

To test for leakage across the condenser, the resistances are left the same as in the previous test. With the galvanometer short-circuited (or open circuited), the vibrator *h* is brought in contact with *a* and left there. On again connecting the galvanometer in the circuit, the current through it should be too small to be observed. In this case both the leakage currents across the vibrator and the one across the condenser flow through the galvanometer. As during the normal operation of the bridge, the vibrator *h* is in contact with *a* only half of the time, and as the bridge is generally so arranged that only half of the leakage current passes through the galvanometer, one may be assured that the leakage across the condenser may increase by a factor of at least 4 before introducing an error in the results.

6. INDUCTANCE OF THE GALVANOMETER COIL

In the derivation of the equations the assumption is made that the inductance of the galvanometer is independent of the current. This is true only for circuits which do not contain ferromagnetic material. Since the galvanometer coil is in close proximity to the magnet, the inductance is not a constant. However, the effect of inductance is merely to decrease the rate at which the condenser charges. Hence, by choosing constants so that the incompleteness of charge is very small, there is little chance that a variation of inductance with current will produce an appreciable effect.

It is difficult to show by direct experiments that the magnetic circuit of the galvanometer has a negligible influence on the charging of the condenser. However, a number of variations have been made without producing any measurable effect on the capacitance. Magnets of varying degrees of hardness have been tried, from a very hard cobalt steel to a soft steel used as an electromagnet. Also the core has been varied through the same range as the magnets. Again iron-cored inductances have been put in series with the galvanometer. In no case was there any indication that the measured capacitance has been affected by an iron-cored inductance in the galvanometer circuit.

7. TIME INTEGRAL OF THE GALVANOMETER CURRENT IS ZERO

In deriving the expression for the capacitance, it was assumed that the integral of the current through a cycle is zero. Expressed symbolically

$$\int_0^T z\,dt = 0$$

However, an observer determines when this condition is realized by observing the position of the galvanometer coil. Hence, it is necessary to determine the relation between the galvanometer deflection and the current through the coil.

The simple galvanometer equation is

$$K\frac{d^2\theta}{dt^2}+M\frac{d\theta}{dt}+U\theta=gHz \tag{56}$$

Multiplying by dt and integrating from 0 to T

$$K(\dot{\theta}_T-\dot{\theta}_0)+M(\theta_T-\theta_0)+U\int_0^T\theta dt=gH\int_0^T zdt \tag{57}$$

If a steady state has been reached, both the velocity and position of the coil are the same at the beginning and end of the cycle, so that the first two terms of (57) are zero. Hence, in order that $\int_0^T zdt=0$ it is necessary that $\int_0^T\theta dt=0$. The problem then resolves itself into determining how the observer may be assured that the time integral of the deflection is zero.

When the Maxwell bridge is in operation the spot of light reflected from the galvanometer mirror onto a ground-glass scale is spread out into a band of light. Not infrequently this band may be a millimeter or more in width with one edge much more intense than the remainder of the band. The observer is inclined to make this bright edge correspond to the zero position. Evidently this is not correct, for the deflection will then all be on one side of zero, and the integral can not be zero. It is therefore important to determine where the setting should be made, or to determine the conditions under which a setting on the bright edge will not introduce an appreciable error.

If the spot of light is photographed on a film moving in a vertical direction, the trace on the film will indicate the motion of the galvanometer coil. Such a photograph is reproduced in Figure 2. The motion of the coil produced on the moving film a series of loops each of which is almost exactly a parabola. As such a film can not be made for each galvanometer and bridge, it is necessary to show theoretically the conditions under which such a curve will be produced and then to determine the proper setting in order that

$$\int_0^T\theta dt=0$$

Under normal conditions of operation of the bridge, the charging of the condenser takes place in an extremely short interval of time. The galvanometer coil behaves as though it were struck a sudden blow at the beginning of each cycle; the torque produced by the steady current through the coil first stops the motion and then returns the coil at the end of the cycle to its original position. Hence, to obtain the equation of motion, it suffices to solve the galvanometer equation

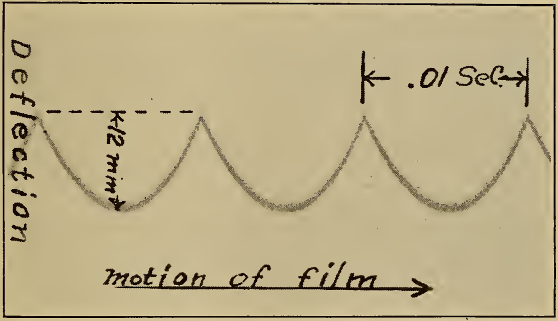

Fig. 2.—*Photograph of the forced vibration of a galvanometer coil used in a Maxwell bridge*

The film moving uniformly in a vertical plane was placed at a distance of 6 meters from the galvanometer. The bridge was used to measure a 0.1 microfarad air condenser. The period of the galvanometer was about 10 seconds

(56) under the assumption that z is a constant. This neglects the first part of the curve when the condenser is being charged. However, in this interval the motion of the galvanometer coil is negligibly small though the change in velocity is relatively large.

The solution of the galvanometer equation when z is a constant is

$$\theta = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \frac{gHz}{U} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \qquad (58)$$

where

$$\lambda = -\frac{M}{2K} \pm \frac{\sqrt{M^2 - 4KU}}{2K}$$

and the A's are arbitrary constants to be determined from the conditions that θ shall be the same at the beginning and end of the cycle and that $\int_0^T \theta dt = 0$. For the type of galvanometer generally used with a Maxwell bridge where the moment of inertia is large and the damping approximately critical, the exponentials in (58) can be expanded in a series, using only a few terms. It follows that

$$\theta = A_1 \left(1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2} + \frac{\lambda_1^3 t^3}{6} + \cdot \cdot \cdot \cdot \cdot \right) + A_2 \left(1 + \lambda_2 t + \frac{\lambda_2^2 t^2}{2} + \frac{\lambda_2^3 t^3}{6} + \cdot \cdot \cdot \cdot \cdot \cdot \right) + \frac{gHz}{U} \qquad (59)$$

$$\theta_T - \theta_0 = A_1 \left(\lambda_1 T + \frac{\lambda_1^2 T^2}{2} + \frac{\lambda_1^3 T^3}{6} + \cdot \cdot \cdot \cdot \cdot \cdot \right) + A_2 \left(\lambda_2 T + \frac{\lambda_2^2 T^2}{2} + \frac{\lambda_2^3 T^3}{6} + \cdot \cdot \cdot \cdot \cdot \right) = 0 \qquad (60)$$

$$\int_0^T \theta dt = A_1 \left(T + \frac{\lambda_1 T^2}{2} + \frac{\lambda_1^2 T^3}{6} + \cdot \cdot \cdot \cdot \cdot \cdot \cdot \right) + A_2 \left(T + \frac{\lambda_2 T^2}{2} + \frac{\lambda_2^2 T^3}{6} + \cdot \cdot \cdot \cdot \right) + \frac{gHzT}{U} = 0 \qquad (61)$$

Solving (60) and (61) for A_1 and A_2, substituting in (59) and simplifying

$$\theta = \frac{gHz}{2K} \left(t^2 - tT + \frac{T^2}{6}\right) \qquad (62)$$

This solution includes all the second-degree terms in the expansion of $e^{\lambda t}$.

Equation (62) is the equation of a parabola. The important condition used in its derivation is that the moment of inertia of the galvanometer coil shall be large. This is equivalent to requiring that the period of the galvanometer shall be long. As a galvanometer of this type is always used in a Maxwell bridge, it follows that the motion of the galvanometer coil can always be considered as a series of parabolas.

Equation (62) can also be used to determine the correct zero position. In deriving this equation the zero of θ was so chosen that $\int_0^T \theta dt = 0$. The largest positive value of θ is when $t=0$ or T; then $\theta = \frac{gHz}{2K} \cdot \frac{T^2}{6}$. The largest negative value of θ is when $t = \frac{T}{2}$; then $\theta = -\frac{gHz}{2K} \cdot \frac{T^2}{12}$. Hence, for correct integration, the band of light should extend twice as far on one side of zero as on the other, with the brighter edge of the band nearer the zero position.

However, the most important use of equation (62) is to show the properties that must be possessed by a galvanometer in order that a setting can be made on the bright edge of the band of light without introducing an appreciable error in the bridge balance. When such a setting is made, the error is one-third of the total width of the band of light or $\frac{\theta_m}{3}$ where θ_m is the width of the band. Now a change in the current z by an amount Δz will produce a change in deflection ($\Delta\theta'$) which is given by the equation $\Delta\theta' = \frac{gH}{U} \Delta z$. The error $\frac{\theta_m}{3}$ introduced by setting on the bright edge of the band of light is found by equating values of $\theta_m/3$ and $\Delta\theta'$. Hence

$$\frac{gHzT^2}{24K} = \frac{gH\Delta z}{U} \qquad (63)$$

Rearranging and inserting the value $T_g = 2\pi\sqrt{\frac{K}{U}}$, where T_g is the undamped period of the galvanometer

$$\frac{\pi^2 T^2}{6T^2_g} = \frac{\Delta z}{z} \qquad (64)$$

Since the error in the capacitance is approximately equal to the error in the steady current z through the galvanometer, and since $\pi^2/6$ is sufficiently near unity for the present discussion, the error caused by setting on the bright edge of the band of light is negligible if the square of the ratio of the period of charge and discharge to the period of the galvanometer is less than the allowable error in the capacitance.

The above conclusion that the integral of the galvanometer current is zero when the integral of the deflection is zero is based on the assumption that the torques acting on the galvanometer coil are sufficiently well represented by equation (56). As a test of this assumption, it is desirable to have an independent experimental method for determining whether or not a galvanometer correctly integrates the current. Such a method is shown diagrammatically in Figure 3. The galvanometer to be investigated is so connected to a condenser and the tongue of a vibrator that both the charging and discharging currents pass through it. If the insulation is good, this set-up assures that the integral of the current is zero. Hence, if the integral of the deflection is zero, the galvanometer integrates correctly.

If the resistance of the charging and discharging circuits is low, then the galvanometer receives a series of blows first in one direction and then in the other. However, the galvanometer can be subjected to conditions which more nearly approach those in the Maxwell bridge by inserting a rather high resistance in either the charge or discharge circuit. If a galvanometer integrates correctly under these conditions, it is probable that it will integrate correctly in a Maxwell bridge. This experimental test on a number of commercial galvanometers showed that in general they did not integrate correctly. By carefully orientating the coil relative to the magnets, any given galvanometer can, if there are no unusual conditions, be made to integrate with high accuracy, but with many galvanometers this adjustment must be made so carefully that they are not suitable for use in a Maxwell bridge. Hence, it seemed desirable to make a theoretical study of galvanometers for this special use.

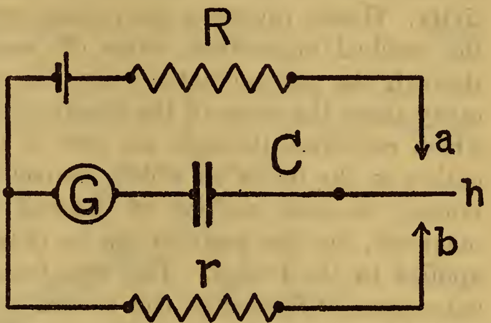

Fig. 3.—*Diagram of a set-up for determining whether a galvanometer integrates correctly*

V. THEORY OF THE GALVANOMETER

In order for a galvanometer to be useful in a Maxwell bridge the time integral of the deflection must be zero when the time integral of the current through it is zero or

$$\int_0^T \theta dt = 0 \text{ when } \int_0^T z dt = 0$$

In deriving the formula, J. J. Thomson assumed that this condition was fulfilled. Although Chrystal,[13] Lord Rayleigh,[14] and Russel [15] early observed that galvanometers when used with a mutual inductance do not always integrate correctly, yet it was only recently that Campbell [16] observed a similar effect in connection with a Maxwell bridge.

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[13] Chrystal, G., "On Bi and unilateral galvanometer deflection," Phil. Mag. (5), 2, p. 401; 1876.

[14] Lord Rayleigh, "On the interpretation of the galvanometer as a test of the evanescence of a transient current," British Assocn. Reports, p. 444; 1883.

[15] Russel, A., "The dead points of a galvanometer needle for transient currents," Phil. Mag. (6), 12, p. 202; 1906; or Proc. Phys. Soc., London, 20, p. 235; 1905-1907.

[16] Campbell, A., Maxwell's Commutator Bridge Method, Glazebrook's Dictionary of Applied Physics, 2, p. 125; 1922.

It has already been shown that a galvanometer will integrate correctly provided the magnetic field H is a constant. There is implied the further assumption that the coil itself is strictly nonmagnetic. The effect of these two assumptions will be considered separately.

1. EFFECT OF MAGNETIC MATERIAL IN THE GALVANOMETER COIL

A galvanometer coil generally contains some ferromagnetic material either in the frame on which the coil is wound or in the insulation of the wire, or in the wire itself. Such material has some retentivity. Hence, reversing the voltage applied to the bridge will reverse the residual magnetism, since the maximum value of the current through the galvanometer when the condenser is being charged is many times the value of the steady current in the opposite direction, which continues through the rest of the cycle. The residual magnetism in the frame on which the coil is wound has a part in determining the zero position of the coil. Hence, if the coil frame is magnetic, the zero position can be changed by reversing the voltage applied to the bridge. This was found to be the case with every galvanometer having its coil wound on an aluminum frame. When used for measuring large condensers where the maximum value of the current through the galvanometer may be an ampere, a zero shift of a centimeter on reversing the current was observed with commercial galvanometers. While the first effect of this can be eliminated by measuring from the zero corresponding to the current direction in use, a galvanometer which shows an appreciable zero shift on reversing the applied potential is not convenient to use. Hence the galvanometer coil should not be wound on a frame which shows any appreciable magnetic properties.

The magnetic character of the wire and insulation appears to be of less importance than that of the frame. In order to obtain high sensitivity, it is important that the permeability of the coil shall be as near as possible to that of the surrounding air. However, there has been no experimental evidence that the integration is affected by the permeability of the coil.

2. EFFECT OF THE GALVANOMETER CURRENT ON THE MAGNETIC FIELD

If the coil of a galvanometer is in an unsymmetrical position with respect to the magnet, as shown in Figure 4, then when a current flows in the coil the total flux through the magnet consists of two parts, one the part associated with the permanent magnetism of the magnet and the other the part caused by the current through the coil. The latter part can be represented by a power series of the current and of the angle θ' which the coil makes with its symmetrical position.

Hence the magnetic field H at the coil, due to the flux through the magnet, is given by the equation

$$H = H_0 + H_1 z + H_2 z^2 + \dots\dots \tag{65}$$

where

$$H_1 = H'_1(\theta' + \alpha\theta'^2 + \dots\dots)$$
$$H_2 = H'_2(\theta' + \beta\theta'^2 + \dots\dots)$$

Substituting this in the galvanometer equation (56)

$$K\frac{d^2\theta}{dt^2} + D\frac{d\,\theta}{dt} + U\,\theta = gH_0 z + gH_1 z^2 + gH_2 z^3 + \dots\dots \tag{66}$$

Multiplying by dt, integrating, applying the condition that the velocity and position of the coil are the same at the beginning and end of the cycle, and assuming that the motion of the coil is so small that H_1 and H_2 can be considered constants, there results the equation

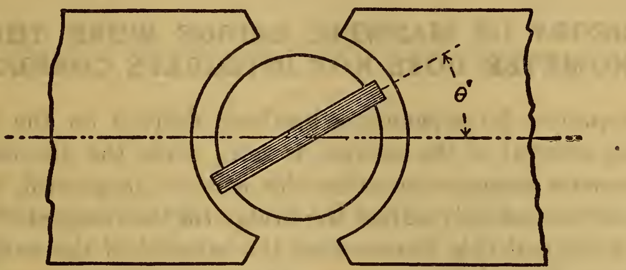

FIG. 4.—*Diagram showing a galvanometer coil in an unsymmetrical position with respect to the magnet*

$$U\int_0^T \theta\,dt = gH_0\int_0^T z\,dt + gH_1\int_0^T z^2 dt + gH_2\int_0^T z^3 dt + \dots\dots \tag{67}$$

Now, as z^2 can never be negative, the integral of z^2 is never zero. Also the integral of z^3 is zero only when the current curve is symmetrical with respect to the time axis. Hence the only condition under which the last two terms vanish is that $H_1 = 0$ and $H_2 = 0$. This occurs only when $\theta' = 0$. It is necessary that these terms vanish in order that $\int_0^T z dt = 0$ when $\int_0^T \theta dt = 0$.

An experimental method has already been described by which the position of the galvanometer coil for correct integration can be determined. However, the difficulty of setting the galvanometer coil so that H_1 and H_2 are sufficiently small can be reduced by so designing the galvanometer magnet that the magnetic flux shall change very little with changes in the external magnetic field. This requires that suitable material be chosen from which to make the magnet, that the magnetic circuit shall be of sufficient length, and that the magnetic treatment at the time of magnetization shall be such as to bring out the desired properties. The most suitable materials now available are the recently developed cobalt steels provided they have received the correct heat treatment. The entire magnetic circuit excepting the gap for the coil should be of this material, and the length of magnetic circuit should be as large as in magnets of ordinary materials. The satisfactory magnetization of the finished magnet can only be accomplished by the application of a magnetic field sufficient to produce practical saturation. After magnetization the magnet should be placed in a small alternating field. This leaves the material in a magnetic condition represented by a point on some portion of a subsidiary hysteresis loop, the slope of which is small (H'_1 approaches zero) and the radius of curvature of which is large (H'_2 approaches zero).

VI. THEORY OF MAXWELL BRIDGE WHEN THE GALVANOMETER DOES NOT INTEGRATE CORRECTLY

The equation for capacitance has been derived on the assumption that the integral of the current is zero, while the discussion of the galvanometer theory shows that this will not, in general, be the case. The observer can only adjust the bridge till the integral of the deflection is zero, and this assures that the integral of the current is zero only when H_1 and H_2 are zero. However, by reversing the bridge current and taking the average of the two computed values of the capacitance, a closer approximation is obtained. To estimate the error of this procedure, it is necessary to consider further the fundamental bridge equations.

In equation (12) the integral of z can no longer be taken as zero, so that instead of equation (13) the value of q_1 must be determined from the equation

$$(PS'+BQ)q_1=EQT-(P'S'-Q^2)\int_0^T zdt \;\ldots. \tag{68}$$

From equation (67) this becomes when $\int_0^T \theta dt=0$

$$(PS'+BQ)q_1=EQT+(P'S'-Q^2)\left[\frac{H_1}{H_0}\int_0^T z^2dt+\frac{H_2}{H_0}\int_0^T z^3dt\right] \tag{69}$$

The value of q_1 as given in equation (19) does not depend upon the correct integration of the galvanometer, since the only mechanical condition involved in its derivation is that the velocity of the coil at the beginning and end of the cycle is the same. This condition is fulfilled whenever a steady state is reached. Eliminating q_1 between (69) and (19), solving for C, and, in the coefficient of the integrals, making the approximation that all the other resistances are negligibly small in comparison to S, the value of C is given by the equation

$$C=\frac{QTF}{PS}+\frac{P'}{PE}\left[\frac{H_1}{H_0}\int_0^T z^2dt+\frac{H_2}{H_0}\int_0^T z^3dt\cdots\cdots\right] \tag{70}$$

Now, if E is reversed, the value of S will have to be changed by an amount δ in order to obtain a balance. In equation (70) the sign of E and of z^3 is changed from positive to negative, but the sign of z^2 is unaltered. Hence, with reversed current, equation (70) becomes

$$C=\frac{QTF}{P(S-\delta)}-\frac{P'}{PE}\left[\frac{H_1}{H_0}\int_0^T z^2dt-\frac{H_2}{H_0}\int_0^T z^3dt\cdots\right] \tag{71}$$

Adding equations (70) and (71), dividing by 2, neglecting squares and higher powers of δ, and putting S_a for the average value of S with direct and reversed current, so that $S_a=S-\frac{\delta}{2}$

$$C=\frac{QTF}{PS_a}+\frac{P'}{PE}\frac{H_2}{H_0}\int_0^T z^3dt \tag{72}$$

Equation (72) shows, that by reversing the applied electromotive force, the effect of the term in H_1 is entirely eliminated, but the effect of H_2 is not eliminated. However, H_2 is generally small, so that the error caused by neglecting the term containing the integral of z^3 is usually, but not always, negligible. Moreover, since the value of z is proportional to the value of the capacitance which is being measured, the cube of z and hence its integral will increase as the cube of the capacitance. It follows that the correction term in (72) is much more important with large condensers than with small ones. This shows that the design of a galvanometer to measure large condensers is much more difficult than that for small ones.

VII. CALCULATION OF THE CAPACITANCE

The capacitance is given by equation (20) as

$$C=QT\frac{(Q+P+G)\ (Q+S+B)-Q^2}{[S(Q+P+G)+QG]\ [P(Q+S+P)+BQ]}$$

Computation by this equation is very laborious. J. J. Thomson put it in the form

$$C=\frac{Q}{nPS}\ \frac{1-\dfrac{Q^2}{(Q+P+G)\ (S+Q+B)}}{\left[1+\dfrac{QG}{S(Q+P+G)}\right]\left[1+\dfrac{BQ}{P(S+Q+B)}\right]}=\frac{QF}{nPS} \tag{73}$$

where $n=1/T$ and F is the compound fraction.

The computation can now be divided into three parts; namely (1) the resistance $\frac{Q}{PS}$, (2) the reciprocal of the frequency $\frac{1}{n}$, and (3) the correction factor F. If the condenser and vibrator are replaced[17] by a standard resistor R which can be varied until the galvanometer deflection is zero, then $\frac{Q}{PS}=\frac{1}{R}$. This reduces the resistance values which must be known from 3 to 1. Also all leads to the resistances are maintained the same as in the original bridge, so that no resistance corrections on this account are required.

The frequency n of charge and discharge must be known with the same accuracy as the resistances. However, this frequency will be determined entirely by the charge and discharge apparatus. The accuracy which can be attained with different types of apparatus will be considered in a later section.

The correction factor F never differs greatly from unity. Hence, as a first approximation

$$F=1-\frac{Q^2}{(Q+P+G)\,(S+Q+B)}-\frac{QG}{S(Q+P+G)}-\frac{QB}{P(S+Q+B)} \qquad (74)$$

If S is large relative to Q and B, as is generally the case, then as a further approximation

$$F=1-\frac{Q}{S}\left[\frac{Q+G}{P+Q+G}+\frac{B}{P}\right] \qquad (75)$$

Of the resistances involved in equation (75), all can be accurately measured except G and B. The resistance G occurs both in the numerator and denominator of the fraction $\frac{Q(Q+G)}{S(P+Q+G)}$ so that the effect of uncertainty in its value is not easily stated. Let the per cent uncertainty in G be represented by δ. Then the uncertainty in the fraction can be determined by substituting for G first $G\ (1+\delta)$ and then $G\ (1-\delta)$, and subtracting the resulting fractions. When terms containing δ^2 are neglected, this gives $\frac{2GPQ\delta}{S(P+Q+G)^2}$ as the uncertainty in the fraction. Let $Q=r\ P$. Since F is approximately unity, $\frac{Q}{S}=nCP$. Substituting in the above fraction the uncertainty becomes $\frac{2nCP^2G\delta}{[P(1+r)+G]^2}$. This fraction is zero if P is zero, and increases with increasing P. However, it does not increase indefinitely but approaches the value $\frac{2nCG\delta}{(1+r)^2}$

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[17] Rosa and Grover, The Absolute Measurement of Capacity, B. S. Sci. Paper No. 10, Bull. Bureau of Standards, 1, p. 153.

when $P\ (1+r)$ is large relative to G. In any case the values of P and Q should be kept small as this decreases the uncertainty in F.

The resistance B of the battery is contained in the term $\frac{QB}{SP}$. Substituting the value of $\frac{Q}{S}$ given above, this becomes nCB. Hence, for a given condenser, measured with a given rate of charge and discharge, the effect of an uncertainty in B is independent of the resistances in the bridge arms. Moreover, since B is largely an electrolytic resistance, in which the uncertainty is always large, a battery should be chosen in which B is small. The most suitable battery is a lead storage battery having a rather large discharge rate. With such batteries, the effect of the uncertainties in B on the value of F is negligible.

Where such batteries are not available, two batteries having as nearly as possible the same characteristics can be used. One of these batteries is first used independently, and the capacitance computed as though the value of B were zero. This gives a value of the capacitance which is too large. The two batteries are then used in parallel, giving a value of the capacitance which lies midway between the value that would be obtained with zero battery resistance and the value obtained with a single battery. Hence, the determinations with a single battery and two identical ones in parallel can be extrapolated to give the value that would be obtained with a battery of zero resistance. This is a more satisfactory method of determining the correction for battery resistance than the insertion of a measured value of the battery resistance in the formula for F.

The computation of F by equation (75) is not difficult, since all the required multiplications and divisions can be performed on a slide rule. However, where there are a considerable number of condensers of about the same value to be measured, it is often convenient to construct a table from which the value of F can be readily taken.

VIII. APPARATUS FOR CHARGING AND DISCHARGING THE CONDENSER

Two distinct types of apparatus have been employed by previous investigators for charging and discharging the condenser; namely, a tuning fork and a rotating commutator. The tuning fork was used by Rayleigh and by Rosa, but most other investigators have used a rotating commutator. Both methods have been used in this investigation.

The tuning fork has the great advantage of a more constant rate than is possible with the rotating commutator. This permits an accurate setting of the bridge, which is a matter of prime importance.

However, there is more difficulty in getting satisfactory contacts for charging and discharging the condenser when using a tuning fork than when using a rotating commutator. On account of this difficulty it has not been feasible to use a fork to give a rate of charge and discharge greater than 100 per second, whereas with a rotating commutator a rate as high as 1,000 per second has been used in this laboratory. With both the commutator and the tuning fork it is necessary to determine the rate of charge and discharge while the balance of the bridge is maintained. The method of doing this, however, is very different in the two cases. The speed of the commutator is determined from a standard chronometer by means of a chronograph. The rate of the tuning fork is determined by reference to a pendulum having a known rate. Hence, it is desirable to give independent descriptions of the two methods.

1. ROTATING COMMUTATORS

Rotating commutators have been in almost continuous use in this laboratory for more than 20 years. During that time many changes both in design and method of operation have been introduced. The following description covers only those commutators now in use and the methods now employed.

(*a*) Description of the Commutators.—Two different commutators, one designed to give 100 charges and discharges per second, the other 1,000, are in use. They represent two different types either of which could be used over a wide range of frequency. Both are mounted on the same shaft, which is driven by a direct-current motor at 1,500 revolutions per minute.

The 100-cycle commutator is a modification of the one described by Rosa and Grover.[18] A drawing of this commutator is given in Figure 5. The modifications consisted in making a more rigid mounting for the brushes, and in placing all of them so that they bear on cylindrical surfaces. The brushes can easily be adjusted so that the contact is good with an absence of chattering. However, the capacitance between the parts is relatively large, requiring a correction of about 30 micromicrofarads to every capacitance measured.

The 1,000-cycle commutator is quite different from the 100-cycle one. It consists of a series of segments of special shape mounted on the cylindrical surface of an insulating drum, together with three coplanar brushes which make contact on these segments. A drawing of the commutator is given in Figure 6. This commutator has all the advantages of the one described above, including high insulation resistance and low contact resistance, with the additional advantage that the capacitance between the parts is small, so that a correction of only 18 micromicrofarads is necessary.

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[18] Bull. Bureau of Standards, 1, p. 172; 1905.

When these commutators were designed, the desirability of having the charging continue during the first half of the cycle was not appreciated. Hence, in both these commutators, the time of charge is the same as the time of discharge. The first type of commutator (100-cycle) can not be altered to meet this requirement, since at some time during a revolution the same portion of every segment makes contact with each brush. However, the second type (1,000-cycle) can be made to meet this requirement by replacing the segments with segments of the shape shown in Figure 7. With these segments, the charging brush would be in contact for a half cycle, as is required by the theory.

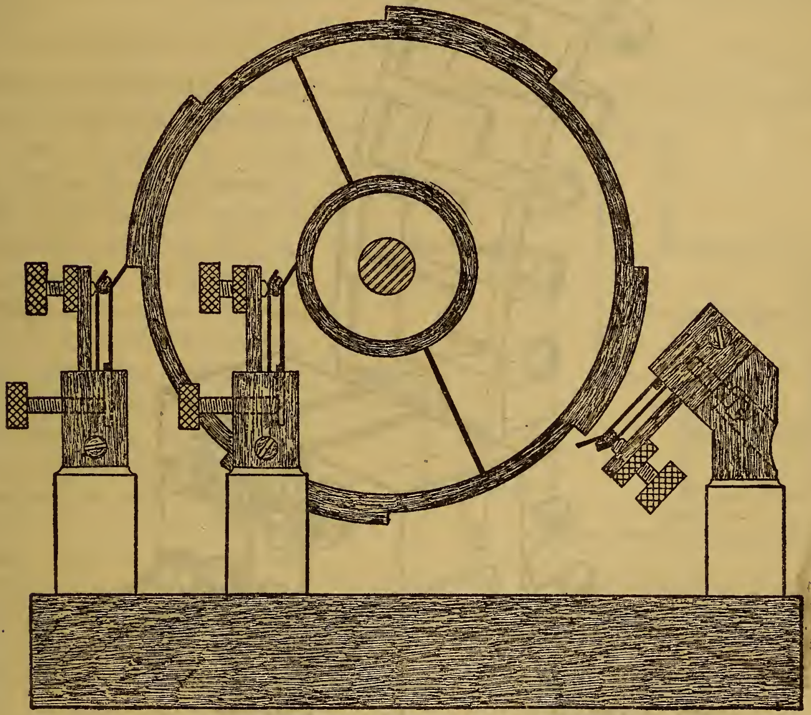

FIG. 5.—*Drawing of the 100-cycle commutator*

(*b*) MEASUREMENT OF THE COMMUTATOR SPEED.—The speed of the commutator is determined by means of the chronograph described by Rosa and Dorsey.[19] The essential features of this chronograph are a drum geared to the commutator and a printing magnet to record seconds on the drum. A convenient size for the drum is 50 cm in circumference and 50 cm long. This printing magnet is moved along a track parallel to the axis of the drum by a screw which is geared to the drum so that the drum makes one revolution while the printing magnet advances 1 mm. The record is made on a sheet of plain paper by the stylus of the printing magnet which strikes a typewriter ribbon interposed between the stylus and the drum. The electric circuit is closed for only a short interval at each second, so that the record on the paper consists of a series of dots. If the speed of the commutator is adjusted so that a revolution of the drum is made in n seconds, then the n^{th} dot will lie 1 mm in the axial direction from the zero dot. Likewise, the second n^{th} dot will lie 1 mm from the n^{th} dot, etc. In such a case, the speed of the commutator can be determined directly from the known gear ratio between the commutator and drum. For instance, if the gear ratio

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[19] Bull. Bureau of Standards, 3, p. 561; 1907.

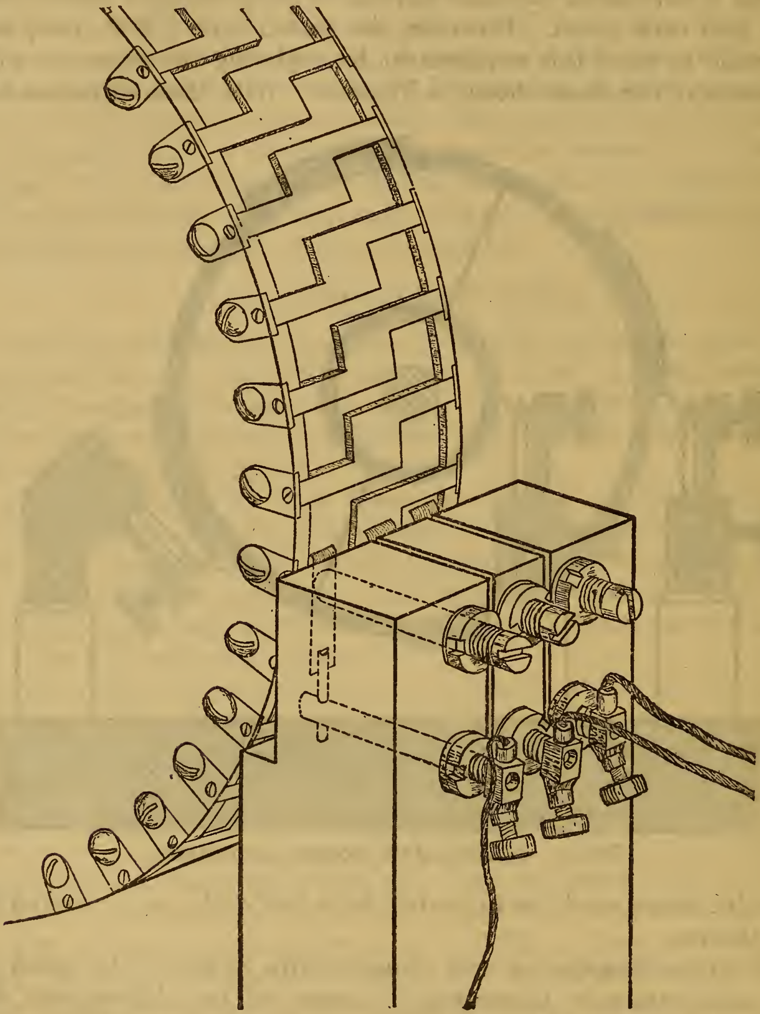

Fig. 6.—*Drawing of the 1,000-cycle commutator*

was 200 and the eighth dot was in the same axial line as the zero dot, the speed would be 200× ⅛=25 revolutions per second.

If the n^{th} dot lies in a line which makes an angle θ with the axial line, the speed of the drum and hence of the commutator can be determined by measuring this angle θ. If the circumference of the drum is x and the distance that the electromagnet travels in the axial direction during one revolution of the drum is y, then the distance that the circumference of the drum travels in n seconds is $x \pm y \tan\theta$. Hence, the number of revolutions per second of the drum is $\frac{1}{nx}(x \pm y \tan\theta)$. Multiplying this by the gear ratio g gives the speed of the commutator as $\frac{g}{n}\left(1 \pm \frac{y}{x}\tan\theta\right)$. All of these are constants except θ, which must be determined from measurements of each record. As used in this laboratory $g=200$, $n=8$, $y=1$, and $x=500$, so that the speed of the commutator is equal to $25 \pm 0.05 \tan\theta$ revolutions per second. With the 100-cycle commutator there are four charges and discharges per revolution, while with the 1,000-cycle commutator there are 40. Hence, the number of charges and discharges per second is $100 \pm 0.2 \tan\theta$ in the first case, and in the second $1{,}000 \pm 2 \tan\theta$. In either case an error of 20′ in θ will introduce an error of 1 in 100,000 in the number of charges and discharges, and hence in the capacitance.

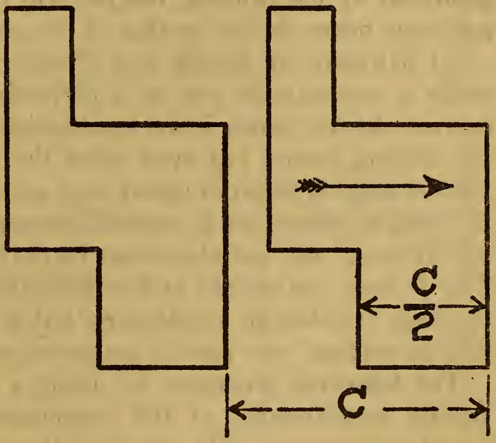

Fig. 7.—*Drawing to show segments of a commutator which will allow charging to continue for a half-cycle*

As the record is made on a sheet of paper which is removed from the drum, it is important that there shall be a record on the paper from which the axial motion can be determined. This is obtained by disengaging the printing magnet from its driving screw, and with the drum running, energizing the electromagnet. As the stylus presses against the paper, a line is made which is in a plane perpendicular to the axis. A line perpendicular to this line gives the axial direction. In practice this last line is not drawn, as the measurement of θ can just as well be made from the original line. Also it is better to make a series of dots by the electromagnet rather than a continuous line. After the paper is removed a straight line is drawn through the dots.

In order to measure θ with the required accuracy, it is necessary that the line of dots should be at least 2 or 3 centimeters long. To obtain a record of this length, the speed must be held constant for three or four minutes. With such a line of dots, the edge of a protractor can be adjusted to it with considerable precision.

In Figure 8 is a reproduction of a portion of a record. Only three of the eight rows of dots are shown.

In the design of the chronograph it is important to so select the reducing gears between the commutator and chronograph that the ratio of the teeth in each pair of gears is an integral number. Then for one revolution of the drum each gear has made an integral number of revolutions, so that at the beginning of each revolution all the gears are meshed in exactly the same way. Hence, the rows of dots produced by the printing magnet will be straight lines even if there are some errors in the cutting of the gears.[20]

(*c*) METHOD OF USING THE COMMUTATORS.—It is impossible to make a commutator run at a perfectly constant speed. A direct-current electric motor is the most satisfactory machine for furnishing the driving power, but even when the motor is driven by a storage battery the variation in speed may amount to several parts in 1,000. By using a tuning fork to control the speed, after the manner described by Wenner,[21] the variations can be reduced to a few parts in 10,000. This is very convenient and sufficiently accurate for the calibration of small variable air condensers, but is more of a hindrance than a help in making very precise measurements.[22]

The following procedure for using a rotating commutator for the precise measurement of the capacitance of a condenser has been found convenient. The speed of the commutator is adjusted until the rows of dots on the chronograph are nearly axial. The bridge is then approximately balanced by varying the resistance *S*, after which the speed of the commutator is slightly increased, which in turn causes the galvanometer to deflect. The galvanometer deflection is then brought back to zero and maintained there by the observer, who at the same time watches both the galvanometer and controls the speed by a slight friction between his finger and the flywheel. In this way the maximum variation in speed will not exceed 1 or 2 parts in 10,000, while the average is very much less.

(*d*) OBSERVATIONAL ERRORS IN USING A ROTATING COMMUTATOR.—The accuracy which can be attained in measuring a capacitance with a rotating commutator depends upon the closeness with which the speed can be maintained constant and upon the exactness with which

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[20] In the train of gears used in making the record shown by Rosa and Dorsey (Bull. Bureau of Standards, **3**, p. 564; 1907) was one pair of gears which had a ratio of 2:5. Hence, the gear errors were eliminated only in every second row of dots.

[21] Phys. Rev. (1), **24**, p. 535; 1907.

[22] The governor described by Giebe in Zeit. für Instr., **29**, p. 205; 1909; has not been used in this laboratory. By means of it, he maintains the speed constant to a few parts in 100,000.

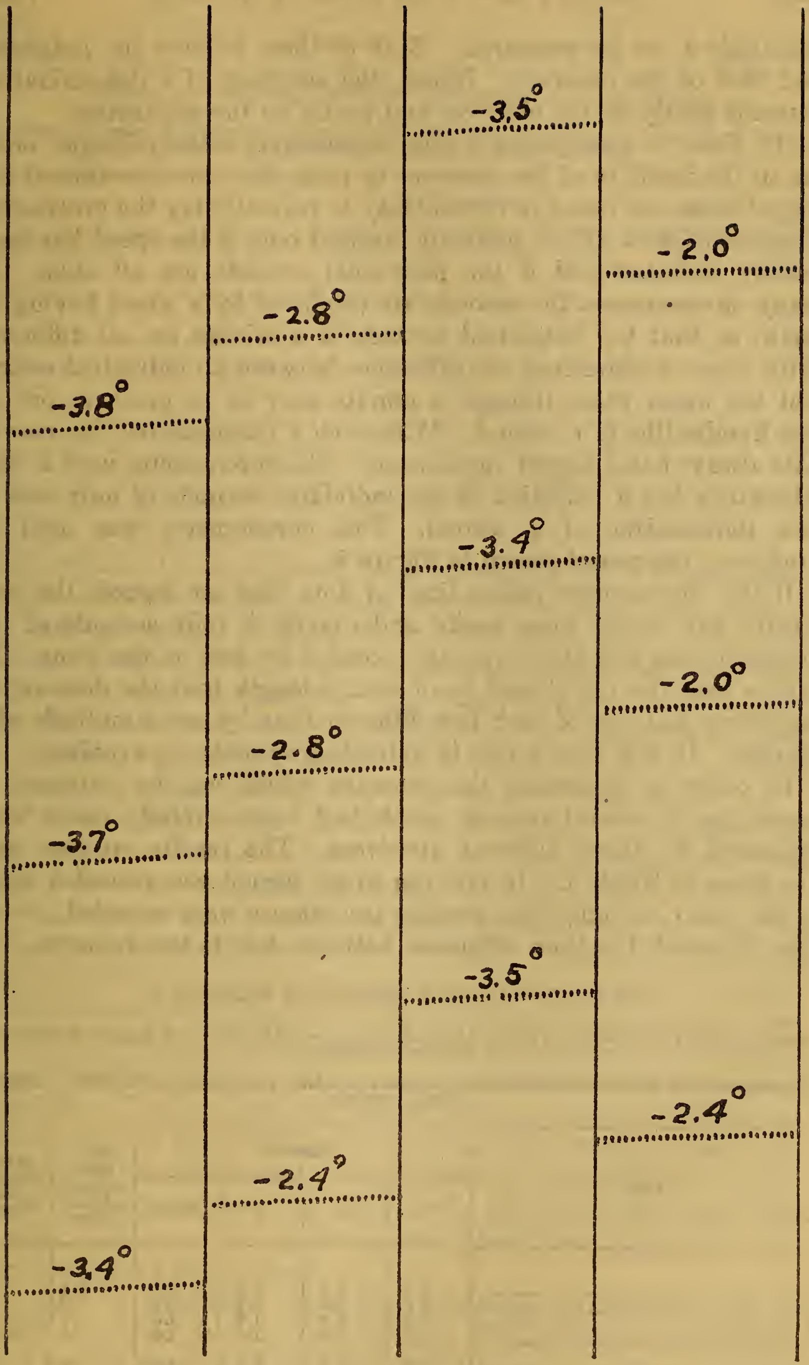

FIG. 8.—*Reproduction of a portion of a chronograph record*

The dots were produced by a magnet actuated by a chronometer. The vertical lines were drawn parallel to the direction of motion of the chronograph paper. The figures give the angles between the rows of dots and a perpendicular to the lines. The minus sign indicates that the speed of the machine to which the chronograph was attached was less than nominal, in this case 25 revolutions per second the angle θ can be measured. Both of these involve the judgment and skill of the observer. Hence, the accuracy of a determination depends partly on the observer and partly on the apparatus.

(1) *Error in determining θ.*—In determining θ the principal error lies in the inability of the observer to judge the true direction of the row of dots, and hence in his inability to correctly lay the protractor. The row of dots will be perfectly straight only if the speed has been exactly constant and if the individual seconds are all alike. In many chronometers the seconds are produced by a wheel having 60 teeth, so that the individual seconds in a minute are all different. With some chronometers the difference between an individual second and the mean value through a minute may be as much as one or two hundredths of a second. With such a chronometer, the row of dots always has a jagged appearance. The chronometer used in this laboratory has a variation in the individual seconds of only one or two thousandths of a second. This chronometer was used in producing the record shown in Figure 8.

If the chronometer makes lines of dots that are jagged, the protractor can be set more easily and exactly if only seconds at the beginning and end of the run are recorded by dots on the drum. In such a case the run should be of such a length that the dots at the beginning and end of each line differ in time by some multiple of a minute. In this way, errors in individual seconds are avoided.

In order to determine the accuracy which can be obtained in measuring θ, several records which had been carefully made were measured by three different observers. The results on two runs are given in Table 1. In one run every second was recorded while in the other run only four seconds per minute were recorded, giving lines in which the time difference between dots is two minutes.

TABLE 1.—*Accuracy obtainable in measuring θ*

[*An error of 0.35° in θ introduces an error of 1 in 100,000 in the speed of the drum, and, hence, in the value of the measured capacitance*]

[Measurements by 3 observers on the 8 lines of a record in which every second is recorded. Length of record, 4 minutes]

Line	Observer				Maximum variation from mean	Mean variation from mean
	C	M	S	Mean		
	°	°	°	°	°	°
1	1.7	2.1	1.4	1.7	0.4	0.23
2	1.7	1.5	1.4	1.5	.2	.10
3	1.3	1.3	1.3	1.3	.0	.00
4	1.4	1.0	1.4	1.3	.3	.17
5	1.8	1.4	1.4	1.5	.3	.17
6	1.5	1.5	1.0	1.3	.3	.23
7	1.4	.9	1.4	1.2	.3	.23
8	1.6	1.8	1.5	1.6	.2	.10
Mean	1.55	1.44	1.35	1.43		.15

TABLE 1.—*Accuracy obtainable in measuring θ*—Continued

[Measurements made by the same 3 observers on the 8 lines of a record in which the dots of any line differ in time by 2 minutes. Length of record, 4 minutes]

Line	Observer				Maximum variation from mean	Mean variation from mean
	C	M	S	Mean		
	°	°	°	°	°	°
1	5.4	5.5	5.3	5.4	0.1	0.07
2	5.8	5.8	5.7	5.8	.1	.03
3	5.4	5.9	5.5	5.6	.3	.10
4	5.9	6.3	6.0	6.1	.2	.17
5	6.0	6.4	6.5	6.3	.3	.20
6	5.8	6.4	6.4	6.2	.4	.27
7	5.4	5.5	6.0	5.6	.4	.23
8	5.4	5.8	5.9	5.7	.3	.20
Mean	5.76	5.95	5.91	5.84	----------	.16

The results do not indicate any superiority for either method. In both cases the maximum difference between the mean value of any observer and the mean of all the observers is only a little over a tenth of a degree, which would introduce an error in the capacitance of about 3 parts in 1,000,000. By either method, using a good chronometer of known rate and continuing the record for at least four minutes, the error in measuring θ will introduce an error in the computed capacitance of only a few parts in 1,000,000.

(2) *Error caused by the inability of the observer to maintain the speed constant.*—In the method of speed control which has been outlined the observer attempts to keep the galvanometer deflection zero—and hence the commutator speed constant—by varying the friction between his finger and the flywheel. However, it has been found impossible to maintain the speed constant to 1 part in 100,000. A skillful observer is quite sure to let the speed get as much as 1 part in 10,000 away from its mean value at a few times during a four-minute run, while there will be an almost continuous variation of a few parts in 100,000. But it is known that if over an interval of time the integral value of the deflection is zero, then the average speed over that interval should be used in computing the capacitance. Hence, it is only necessary that the observer so maintains the speed that the time integral of the galvanometer deflections, taken over the time of making a run, is zero.

In order to determine with what accuracy an observer can so regulate the speed that the time integral of the galvanometer deflection is zero, a number of sets of runs were made by three different observers. In each set the capacitance and all resistances were kept constant, the only difference being in the average speed as determined by the different observers. The values for θ for a typical set are given in Table 2.

TABLE 2.—*Values of θ as determined from successive runs by different observers*

[*A difference of 0.35° means a difference of 1 in 100,000 in the computed capacitance*]

Observer	θ	Variation from mean
	°	°
M	5.81	−0.17
C	6.42	+.44
S	5.90	−.08
S	6.13	+.15
C	6.08	+.10
M	5.57	−.41
Mean	5.98	

Maximum variation from mean, 0.44°.
Mean variation from mean, 0.17°.

The maximum variation of any value from the mean of all the values is 0.44°, which will cause an error in the computed capacitance of a little more than 1 part in 100,000.[2] This and other similar sets indicate that the error in the capacitance due to the failure of the observer to correctly maintain the speed should be less than 2 parts in 100,000. From this the conclusion can be drawn that in using a rotating commutator for the absolute measurement of capacitance the error introduced by a failure to keep the speed constant can be made less than a part in 100,000 by taking the average of a number of runs.

2. VIBRATION COMMUTATOR

Only one type of vibration commutator has been used in this investigation. On the prongs of a vibrating tuning fork are mounted platinum wires that make contact with mercury cups during a part of each cycle. The vibration of the fork is maintained by an electron tube drive. Such a commutator charges and discharges the condenser with such regularity that settings on a Maxwell bridge can be made with the same ease and certainty as on a Wheatstone bridge. However, the rate of a fork, while very uniform over short intervals of time, is so affected by temperature, by the reaction with the driving circuit, and by the contacts for charging and discharging the condenser that appreciable variations occur from day to day and even from hour to hour. On account of these variations a method was devised for measuring the rate of the fork at the same time that the bridge is being balanced.

(*a*) TUNING FORK AND ITS DRIVE.—A photograph showing a tuning fork which has been successfully used in this laboratory is given in Figure 9. The prongs are approximately 33 cm long, 2 cm wide, and 1½ cm thick. Two forks, the dimensions of which are almost identical, have been used, one made from annealed tool steel, the other from the nickel-iron alloy known as elinvar. In each case the frequency is approximately 100 cycles. A double amplitude of 4 mm at the end of the prongs can be obtained. The principal difference between the two forks is their temperature coefficient. This has

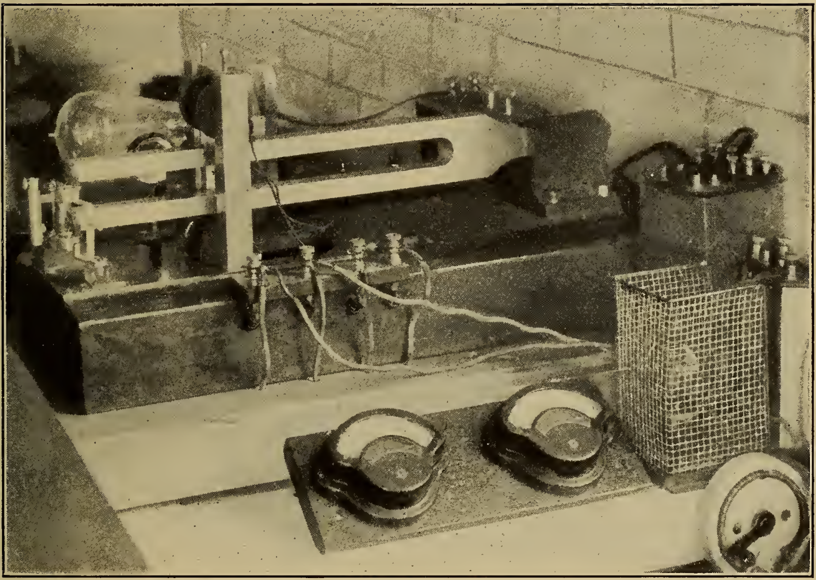

FIG. 9.—*Photograph of the tuning fork used as a vibration commutator*

The apparatus used in connection with the electron-tube drive is shown in the foreground. In the background is the lamp and lens used in rating the fork. Near the end of the prongs are the mercury cups in which contact is made by platinum wires attached to the fork

not been measured in this laboratory. However, a number of observers have reported 110 parts per million per degree centigrade for a steel fork, whereas the coefficient for the elinvar fork is given by the makers as 14 parts per million.

The fork is driven by an electron tube using a circuit similar to that described by Eckhardt, Karcher, and Keiser.[23] To insure a uniform drive and hence a constant frequency, both the filament current and the plate voltage are maintained as constant as possible by the use of storage batteries. The driving coils and their mountings were designed for the steel fork. However, no difficulty was experienced in driving the elinvar fork with the same coils though some adjustments in the electrical circuits were necessary.

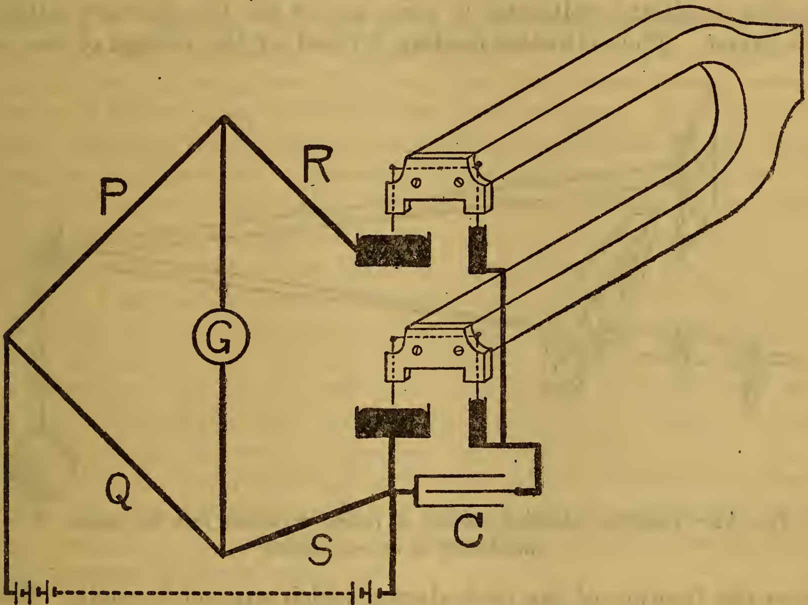

FIG. 10.—*Diagram to show method of operation of the vibration commutator*

(*b*) ELECTRICAL CONTACTS.—The only contacts which have been used are platinum against mercury. The arrangement of these contacts in connection with the tuning fork and bridge is shown in Figure 10. On each prong of the fork is mounted a small block of insulating material. In each block are set, in a vertical position, two platinum wires which are connected together at their upper ends. Underneath each of the four platinum wires is a mercury cup, there being a large and a small cup for each prong. The mercury level is so adjusted that when the fork is vibrating the platinum wires make continuous contact with the mercury in the small cups, while with the large cups contact is made only part of the time. Since the two prongs of a tuning fork vibrate in synchronism but with a phase difference of 180°, the contacts of the upper prong are closed when those of the lower are opened, and vice versa. The mercury level should be so adjusted in the large upper cup that contact will be made for half of a vibration, thus opening the condenser circuit when the velocity of the galvanometer coil is zero. In the large lower cup contact should be made for less than half of the cycle. The adjustment of these levels must be made with the fork running, since standing waves on the surface caused by the vibrations of the fork will affect the contact. The adjustment can be tested by connecting a cell and voltmeter in series across the two mercury cups of one prong. The voltmeter reading divided by the voltage of the cell gives the fraction of the cycle during which contact is made. With a suitable galvanometer, this adjustment need be only approximate.

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[23] Eckhardt, Karcher, and Keiser, J. Opt. Soc. of Am. and Rev. of Sci. Instruments, 6, p. 949; 1922.

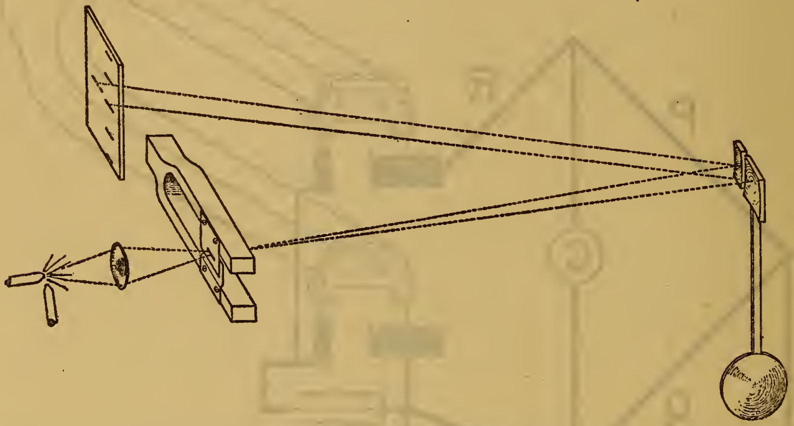

Fig. 11.—*Diagram showing method of rating a tuning fork by means of a pendulum of known period*

In the cups where contact is continuous there is little tendency for the mercury to splash. Hence, these cups can be small, which has the advantage of reducing the capacitance to earth. However, in the cups where the wires leave the mercury considerable splashing does occur. These cups must be relatively large, so that the few mercury droplets which fall outside the cup do not appreciably affect the mercury level in the cup.

(c) Method of Rating the Tuning Fork.—A modification of Reed's method [24] for rating a tuning fork by comparison with a freely vibrating pendulum has been used. A diagram showing the optical arrangement is given in Figure 11. The fork is equipped with a shutter [25] which opens twice during each cycle. The shutter opening is a slit parallel to the axis of the fork. Images of the illuminated slit are formed by the concave mirror mounted on the pedulum.

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[24] Phys. Rev. (I), 12, p. 279; 1901.

[25] This shutter is described by Curtis and Duncan, B. S. Sci. Papers (No. 470), 19, p. 17; 1923.

A screen is so placed as to render visible several images of the slit when the pendulum is near the center of its swing. On the screen is a fiducial line, which may be simply a line drawn parallel to the slit images, or, better, as shown in the figure, an image of the slit from a stationary mirror. For use in a lighted room the images can be observed by a low power eyepiece without the intervention of a screen.

In a given determination of the rate of the fork all the observations on the slit images must be made when the pendulum is moving in one direction. The ease of making observations is greatly increased by swinging across the beam of light an auxiliary pendulum having approximately the same period as the first but with its phase and position so adjusted that it occults every alternate series of images.

If the fork makes an integral number of vibrations while the pendulum is making one vibration, then the observer sees successive series of flashes lying in exactly the same position. In this case the rate of the fork is the frequency of pendulum multiplied by this integral number of vibrations. However, if the number of vibrations which the fork makes during one vibration of the pendulum differs slightly from an integral number, then the observer sees successive series of flashes progressing either upward or downward. He then notes when a flash coincides with the fiducial mark and counts the number of swings of the pendulum until the next flash in the series coincides with this mark. In this time the fork has gained or lost one-half of a vibration. To insure still greater accuracy, the observer starts a stop watch at the time that a coincidence occurs. Several minutes later he stops the watch when another coincidence occurs. In this time both the pendulum and fork have, within the limits of observation, made an integral number of vibrations. To derive the formula for computing the rate of the fork from these observations the following nomenclature is used: Let

p = time for one vibration of the pendulum.

t = time for one vibration of the fork.

$n=\frac{1}{t}$ = number of vibrations of the fork per second.

f = number of vibrations of fork in one vibration of the pendulum expressed to the nearest integer only—known from some previous measurement.

m = number of vibrations of pendulum required for the fork to gain or lose one-half vibration—called the coincidence interval.

b = time required for a complete observation during which the fork makes z vibrations—experimentally determined by a stop watch.

q = number of vibrations of pendulum in time b.

r = nearest integer to q/m = number of half vibrations of fork gained or lost during the time of an observation.

The time b is equal to the number of vibrations of either the pendulum or fork multiplied by the respective period. Hence,

$$qp = tz = t\left(qf \pm \frac{r}{2}\right) \tag{76}$$

Inserting n for $1/t$ and solving

$$n = \frac{1}{p}\left(f \pm \frac{r}{2q}\right) = \frac{f}{p} \pm \frac{r}{2b} \tag{77}$$

The accuracy with which n can be determined depends on the accuracy with which the four constants on the left of equation (77) can be obtained from the experimental observations. Of these constants, the most important is p, the period of the pendulum. Fortunately this can be measured with high accuracy as will be discussed later.

The value of f can generally be determined from the value of the frequency furnished by the makers of the fork. However, a convenient method of measuring f is to use the fork in a Maxwell bridge with a condenser of known capacitance. Then the capacitance formula can be used to compute the frequency.

The value of the integer r is determined either by counting the number of coincidences which occur during an observation or by computing the number of coincidences from the ratio q/m. Since q is relatively large, no difficulty is experienced in determining a value for it which will not introduce an error in r. However, m is small, so that more care must be exercised in determining its value. Counting the number of swings of the pendulum between two coincidences does not necessarily give a sufficiently accurate value. However, if the number of swings is counted over several coincidences, the value of m can be obtained with the required accuracy. As an example, the number of swings between two successive coincidences was about 6, and the number of swings q between two good coincidences (determined from the time obtained by a stop watch) was 53. From this one would determine r as 9. However, by counting over two coincidences, 13 swings were observed which gave $m = 6.5$. From this $r = 8$. A further count over 4 coincidences gave 27 swings or $m = 6.75$, and again $r = 8$. Hence the value of $m = 6$ was not sufficiently accurate, but either of the other values gave a correct value for r. A safe rule is to count at least three times as many swings as the value of r.

The error in an experimental determination depends solely on the error in b. This error arises not so much from errors in the stop watch as from the inability of the observer to judge on which swing of the pendulum the coincidence occurs. This latter is particularly the case if the coincidence interval is long, which is desirable for easy and accurate observation. The error in the observation time, b, may be relatively large, often amounting to the time of one or more swings of the pendulum. It will now be shown under what conditions this observational error produces a negligible error in n.

To find the error in n caused by an observational error in b insert for b in (77) the value $b+\alpha$, where α represents the observational error in seconds. Then, if β represents the error in n

$$n+\beta=\frac{f}{p}\pm\frac{r}{2(b+\alpha)}=\frac{f}{p}\pm\frac{r}{2}\left(\frac{1}{b}-\frac{\alpha}{b^2}\right) \text{ approx.} \qquad (78)$$

Subtracting (77) from this and inserting the value for r

$$\beta=\pm\frac{r\alpha}{2\,b^2}=\pm\frac{\alpha}{2pmb} \qquad (79)$$

As an example, if with a half-second pendulum the fork gains 8 half-vibrations in 26.5 seconds, which may, however, be in error by 0.5 second (one swing of the pendulum), then the error β would be nearly 0.003 of a fork vibration.

Equation (79) can also be used to determine the time over which an observation must extend in order to obtain a desired accuracy. As an example, consider that p, the period of the pendulum, is one-half second; m, the number of pendulum vibrations between coincidences, is 40; α, the observational error, is not more than 1 second, a value which would seldom be exceeded; and β, the allowable error in the frequency of the fork, is 10^{-4}, of a vibration, which corresponds to an accuracy of 1 part in 1,000,000 for a 100-cycle fork. Substituting values in (79) and solving, $b=250$. This shows that under the conditions assumed (which can easily be obtained in practice) the frequency of a 100-cycle fork can be obtained with an accuracy of 1 part in 1,000,000 by observations extending over about four minutes.

Formula (77) is ambiguous in that the sign of the last term is not definite. However, the data obtained in determining f are generally sufficiently accurate to show whether n is greater or less than f/p. If this is not the case, the sign of r/b can be obtained by observing images of the slit, provided an auxiliary pendulum cuts out alternate sets of images. If in each set the flashes sweep up from the bottom to the top of the screen, the pendulum is swinging toward the screen. Then if the flashes in successive sets advance up the screen, the pendulum makes more than a complete vibration during an integral number of vibrations of the fork. Hence, in one swing of the pendulum the number of vibrations of the fork is less than f, so that the sign of r/b is negative. From this the general rule can be drawn that if the direction of progression is, in successive sets, the same as the direction of appearance in each set, the sign of r/b is negative; but if these directions are opposite, the sign is positive.

As previously stated, the value of the frequency n of the fork is no more accurate than the value of the period p of the pendulum. The period of a freely vibrating pendulum can be measured with high accuracy [26] and can be depended upon to remain constant under constant conditions. Three different pendulums were used. A bronze half-second pendulum of the Coast and Geodetic Survey designed for gravity determinations, the second a quarter-second pendulum of design similar to the above, and the third a quarter-second pendulum made from a single piece of fused quartz. The pendulums were mounted in air-tight cases. The pressure in the case and the amplitude of swing were always so nearly the same that no corrections on their account were necessary. With the bronze pendulum a correction for the temperature was necessary. With the quartz pendulum, this correction was negligibly small. In all cases p was known to 1 or 2 parts in 1,000,000.

IX. EXPERIMENTAL TEST OF THE METHOD

In order to determine experimentally the accuracy that can be obtained in the absolute measurement of capacitance, two entirely independent Maxwell bridges were set up so that they could be used to measure the same capacitance. The measurements on the two bridges were referred to the same resistance and to time from the same chronometer. The bridges themselves were entirely distinct, using different resistances throughout, different galvanometers, and different batteries. Moreover, the rotating commutator was used on one bridge (designated as the old bridge) while the vibration commutator was used on the new bridge. With the old bridge the balance was made by adjusting the speed and determining the frequency of charge and discharge by means of a chronograph. With the new bridge the adjustment was made by a resistance at the same time determining the rate of vibration of the commutator by comparison with a pendulum. Hence, conditions were made as different as possible.

In using these bridges over a period of several months, much valuable information has been collected, the important facts of which have been embodied in the preceding pages of this paper. Many anomalous behaviors have not as yet been explained. Imperfect insulation, variable resistance at the commutator contacts, and uncertainty in battery resistance are some of the things that sometimes cause trouble.

As an example of what may be obtained under favorable conditions, the data obtained on one day are given in Table 3.

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[26] The method of measurement of period of pendulum used in this laboratory is based on that described by Swick in "Modern Methods for Measuring the Intensity of Gravity," Special Publication No. 69 of United States Coast and Geodetic Survey. This publication also describes the half-second gravity pendulum.

TABLE 3.—*Accuracy obtained in the measurement of an air condenser by two independent bridges*

1. *Data common to the two bridges:*
   - Condenser .... 0.25 microfarad air condenser.
   - Reference resistance .... Twenty 2,000-ohm coils in series.
   - Time measurements referred to a Negus chronometer.
2. *Data of the individual bridges:*

Part of bridge	Old	New
Ratio coils	100-ohm Wolff coils	100-ohm L. & N. coils (oil immersed).
S arm	L. & N. box, Curtis coils	L. & N. box; bifilar coils.
Galvanometer	Remodeled Sullivan, resistance 5.7 ohms.	Special—resistance 11.4 ohms.
Battery	50 volts from 12 trays of small cells, in parallel.	50 volts from a 24-ampere hour battery.
Commutator	Rotating	Vibration.

3. *Corrected values of capacitance in microfarads.*

Time	Observer	Old bridge	New bridge
2 p. m.			0. 249241
2.45	C	0. 249235	
3.10	M	. 249239	
3.25	S	. 249235	
3.50			. 249244
Mean		. 249236	. 249242

The values obtained by means of the new bridge show a gradual increase in the capacitance, amounting to 1 part in 100,000 during the two hours of the observation. This can readily be explained by changes in the room temperature. The values obtained by the old bridge are consistent among themselves, the maximum variation from the mean being only 1 part in 100,000. However, there is a difference between the average for the two bridges of nearly 2½ parts in 100,000. This indicates that there is some systematic error. Another set of data (less complete) gave practically identical values for the two bridges, and still another gave about the same difference as shown in Table 3 but with the order for the two bridges reversed. Measurements on a 0.1 air condenser showed an error of about the same magnitude. Hence, even with well-designed and carefully adjusted apparatus, there may be an error as large as 2 or 3 parts in 100,000 when a capacitance of 0.1 microfarad or larger is measured by Maxwell's method.

In conclusion the authors wish to express their appreciation of the help they have received from their contemporaries. Particular mention should be made of Miss C. Matilda Sparks, who has aided in the measurements and criticized the manuscript.

APPENDIX I. DETERMINATION OF LEAD CAPACITANCE

The lead capacitance is generally determined by disconnecting the lead from the unshielded terminal of the condenser (the position being changed as little as possible) and measuring the capacitance of the leads and commutator as though they were a small condenser. This has two disadvantages: (1) It may disturb the potential distribution in the bridge so that the lead capacitance as measured may be different from that in use, and (2) very high resistances may be required in order to obtain a balance. These disadvantages are avoided by using a suitable resistance in parallel with the leads.

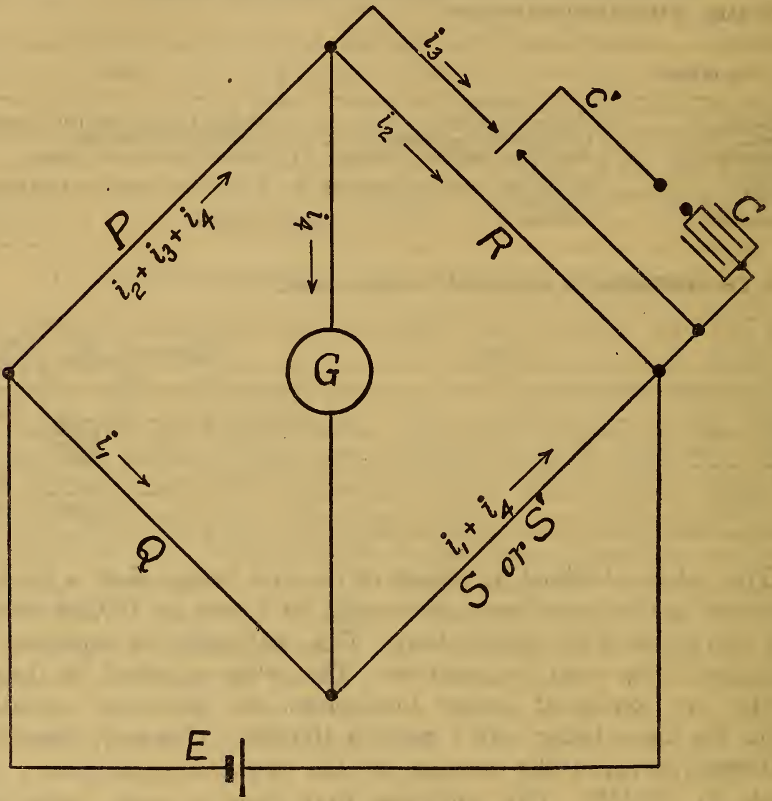

FIG. 12.—*Diagram of a bridge for measuring the lead capacitance by using a resistance in parallel with it*

The lead capacitance C' is disconnected from the main condenser C. A resistance R is connected in parallel with the commutator, which charges and discharges the lead capacitance. When the commutator is at rest or disconnected, the resistance in the lower right-hand arm is S; when the commutator is charging and discharging C', the resistance is S'.

The diagram of connections for using a resistance in parallel with the leads is shown in Figure 12.

The experimental procedure consists of two parts. First, either stop the commutator or disconnect at the junction between P and R and balance the resistances as a Wheatstone bridge. Then

$$PS = RQ \tag{80}$$

Second, start the commutator, insert C' (the lead capacitance to be measured), and balance by adjusting S. Let the new value of S be S'. The value of the capacitance C' is then determined by the equation:

$$C' = \frac{S-S'}{nRS'} \tag{81}$$

where n is the frequency with which the commutator charges and discharges the lead capacitance.

To derive formula (81) apply Kirchoff's laws to the three upper circuits of Figure 12, in which both the direction and magnitude of the current in each arm at a time, t, is indicated. The zero time is taken as the instant that the vibrator touches the left-hand contact. Also, it is assumed that the experimental conditions outlined in the regular derivation of the capacitance formula have been fulfilled. There results the three equations:

$$Qi_1 - Gi_4 - P(i_2 + i_3 + i_4) = 0 \tag{82}$$

$$S'(i_1 + i_4) - Ri_2 + Gi_4 = 0 \tag{83}$$

$$Ri_2 - \frac{1}{C'}\int_0^t i_3 dt = 0 \tag{84}$$

The assumption will now be made that the condenser C' is so small that it charges instantly. Hence,

$$\int_0^t i_3 dt = q$$

where q is the complete charge on the condenser and t has any value greater than zero. Substituting this in (84) and eliminating i_1 and i_2 between the three equations

$$\frac{(PS'-RQ)q}{RC'} + PS'i_3 + (PS' + GS' + QS' + GQ)i_4 = 0 \tag{85}$$

Multiplying by dt', integrating from 0 to T, and inserting the values of the integrals

$$\int_0^T i_4 dt' = 0 \text{ and } \int_0^T i_3 dt' = q$$

$$\frac{(PS'-RQ)Tq}{RC'} + PS'q = 0 \tag{86}$$

Inserting the value of RQ from (80) and solving for C'

$$C' = \frac{T(S-S')}{RS'} = \frac{S-S'}{nRS'} \tag{87}$$

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